On the discrete spectrum of linear operators on Banach spaces

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Introduction

While the spectrum of selfadjoint linear operators in Hilbert spaces is extensively studied, this is not true for linear operators in Banach spaces. This may be best expressed with the words of Davies ([4], Preface):

Selfadjoint operators on Hilbert spaces have an extremely detailed theory. (...) Studying non-selfadjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.

The aim of this thesis is to provide assertions on the discrete spectrum of linear operators on Banach spaces.
We will explain the problem in a little bit more detail.
Let L_0 denote a bounded operator and K denote a compact operator, both defined on a complex Banach space X.
We are interested in the discrete spectrum of the bounded operator

$$L := L_0 + K.$$

In this thesis we want to clarify the following questions:

- Find upper bounds for the number of eigenvalues in some regions of the resolvent set $\rho(L_0)$ of the unperturbed operator.
- What is the rate of accumulation of the discrete spectrum to the essential spectrum?
- What are the possible points of accumulation of the discrete spectrum in the essential spectrum?

This will be done with methods of complex analysis. That means we construct holomorphic functions on domains $\Omega \subseteq \rho(L_0)$ which zeros coincide with the discrete spectrum of L and which have useful upper bounds.

To explain the word useful, let us assume that $\Omega \subseteq \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is a simply connected component of the resolvent set of L_0 with $\infty \in \Omega$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function with the property that $\lambda \in \Omega$ is a zero of order m of f if and only if λ is a discrete eigenvalue of L with algebraic multiplicity m. Additionally, assume that there is a $C > 0$ such that

$$
\begin{aligned}
|f(z)| \leq C & \text{ for all } z \in \Omega \\
\text{and } & |f(\infty)| = 1.
\end{aligned}
$$

Since Ω is simply conncected there has to be a conformal map $\phi : \Omega \to \mathbb{D}$ with $\phi(\infty) = 0$. Then $g := f \circ \phi^{-1}$ is a holomorphic function on the open unit disc and there is a one to one correspondence between the zeros of g and the discrete eigenvalues of L in Ω. Hence, one can apply the Jensen identity (see e.g. Rudin [33] p. 308) and it is possible to derive an estimate for the number of eigenvalues in sets $\Omega' \subseteq \Omega$

$$
\mathcal{N}_L(\Omega') := \#\sigma_d(L) \cap \Omega' \leq \frac{1}{\log \frac{C}{r_\Omega(\Omega')}}
$$

with $r_\Omega(\Omega') < 1$[1].
We will discuss Jensen's identity in Section 4 and the above eigenvalue inequality in Section 5.
This is only a simple example which can occur. In general we are only able to obtain non constant bounds for f, exploding when $\partial\Omega$ getting close to $\sigma(L_0)$.
In fact, if X is a Hilbert space and L_0 a selfadjoint operator with $\sigma(L_0) = [a, b]$ and $K \in \mathcal{S}_p(X)$ is in some Schatten von Neumann class we can construct, using regularized determinants, a holomorphic function d on the whole resolvent set of L_0 with

$$
|d(z)| \leq \exp\left(\Gamma_p \|K(z\mathbb{1} - L_0)^{-1}\|_{\mathcal{S}_p}^p\right).
$$

From this inequality the following Lieb-Thirring type inequality is derived:

$$
\sum_{\sigma_d(L)} \frac{\operatorname{dist}(\lambda, [a, b])^{p+1+\tau}}{|b - \lambda||a - \lambda|} \leq C(p, \tau)(b - a)^{-1+\tau} \|K\|_{\mathcal{S}_p}^p
$$

with $0 < \tau < 1, p \geq 1 - \tau$, where $C(p, \tau)$ is a constant depending only on p, τ. This has already been done in [6] p. 132.
In this thesis we derive similar results for Banach spaces under the assumption that K is an element of some quasi-Banach ideal $(\mathfrak{B}_p, \|\cdot\|_{\mathfrak{B}_p}, c_{\mathfrak{B}_p})$ which

[1] $r_\Omega(\Omega') := \sup_{\omega \in \Omega'} |\phi(\omega)|$

is assumed to be one natural generalization of the Schatten von Neumann Banach ideal, e.g. the space of nuclear operators, linear maps of type l^p or $p-$summing operators.
Moreover, there are obtained several results only under the assumption that K is the uniform limit of finite rank operators.
If L is a selfadjoint bounded operator then the spectrum of L is real. In case that the essential spectrum is an intervall $[a, b]$ then the discrete eigenvalues cannot accumulate in any point of (a, b).
Such an assertion is not possible if L is just an operator acting on a Banach space. But we will also give similar answers which will be strongly connected with the identity theorem of complex analysis.
In the following the content of this thesis is briefly summarized.
Chapter 1 serves as an introduction for basic notations like resolvent sets, spectrum or compact operator and common results about these terms.
In Chapter 2 there will be discussed different typs of quasi-Banach ideals. In particular the eigenvalue distributions of operators belonging to these ideals are of interest.
For each quasi-Banach ideal introduced in Chapter 2 special kinds of determinants are constructed in Chapter 3. One essential property of these determinants is, that they have to provide holomorphicity, i.e. these determinants applied to each holomorphic family belonging to the underlying domain has to be holomorphic by itself. Chapter 4 deals with results in complex analysis. A classical result is the Jensen Identity, a tool counting the zeros of holomorphic functions on the open unit disc.
The preperations of the previous chapters will be applied in Chapter 5 to very general operators $L = L_0 + K$ to obtain results on the number of eigenvalues or on the closure of the discrete spectrum.
In the bounded component of the resolvent set of the free operator L_0 there occur surprising situations after perturbations (e.g. the bounded component changes to pure point spectrum). Chapter 6 is devoted to this topic.
Chapter 7 treats three concrete examples. i.e. the discrete Laplacian on $l^q(\mathbb{Z})$, the operator of multiplication on $C[\alpha, \beta]$ and the shift operator on $l^q(\mathbb{Z})$.
In Chapter 8 there is an attempt to construct holomorphic functions the zeros of which coincide with the discrete spectrum, using perturbation determinants of infinite order on finite dimensional spaces. Finally, in the last chapter some open problems in the context of the results presented in this thesis will be discussed.

Chapter 1

Preliminaries

In this chapter some notations and some basic concepts like spectrum, compact operators and operator ideals are provided.

1.1 Bounded operators, spectrum and resolvents

In the following X denotes a complex Banach space and $\mathcal{L}(X)$ the algebra of all **bounded operators**. The **identity** operator in $\mathcal{L}(X)$ is denoted by $\mathbb{1}$ ($\mathbb{1}f = f$ for all $f \in X$).
We say a bounded operator B is **invertible** iff there exists a bounded operator A with $AB = BA = \mathbb{1}$. In this case $B^{-1} := A$ denotes the **inverse operator** of B.
For any operator $B \in \mathcal{L}(X)$

$$\rho(B) := \{\lambda \in \mathbb{C} : (\lambda\mathbb{1} - B) \text{ is invertible}\}$$

defines the **resolvent set** of B.
Hence, the operator-valued map $\rho(B) \ni \lambda \mapsto R_B(\lambda) := (\lambda\mathbb{1} - B)^{-1}$ is well defined and is called the **resolvent** of B. Moreover, if we fix $\mu \in \rho(B)$ then for every $\lambda \in \{z \in \mathbb{C} : |z - \mu| < \frac{1}{\|R_B(\mu)\|}\}$

$$R(\lambda) := \sum_{k=0}^{\infty} (\mu - \lambda)^k R_B(\mu)^{k+1} \tag{1.1}$$

exists, is bounded and $R(\lambda)(\lambda\mathbb{1} - B) = (\lambda\mathbb{1} - B)R(\lambda) = \mathbb{1}$. This implies $\lambda \in \rho(B)$ and $R(\lambda) = R_B(\lambda)$.
The **spectrum** is defined by $\sigma(B) := \rho(B)^c$.

For any $\lambda \in \sigma(B)$ it holds that $|\lambda| \leq \|B\|$.
As a consequence we have the following corollary.

Corollary 1.1 *Let B be a bounded operator.*

(i) The resolvent set is open,

(ii) the spectrum is compact,

(iii) the resolvent is an analytic map[1]*.*

Remark 1.2 If $|\lambda| > \|B\|$ then the resolvent can be rewritten as

$$R_B(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} B^k,$$

the so-called von Neumann series, which shows that $\lim_{|\lambda| \to \infty} \|R_B(\lambda)\| = 0$. Hence, $R_B(\cdot)$ is analytically extendable to $\hat{\rho}(B) := \rho(B) \cup \{\infty\}$ with $R_B(\infty) = 0$.

The resolvent plays an important role for the analysis of the spectrum, so in the next proposition some more important facts about resolvents are summarized:

Proposition 1.3 *Let A, B be two bounded operators. Then*

(i) $R_B(\lambda) - R_B(\mu) = (\mu - \lambda) R_B(\lambda) R_B(\mu)$ *(first resolvent identity),*

(ii) the resolvent has derivatives of all order with

$$\frac{d^n}{d\lambda^n} R_B(\lambda) = (-1)^n n! R_B(\lambda)^{n+1},$$

(iii) $R_A(\lambda) - R_B(\lambda) = R_A(\lambda)(A - B) R_B(\lambda)$ *(second resolvent identity),*

(iv) $\|R_B(\lambda)\| \geq \frac{1}{\operatorname{dist}\big(\lambda, \sigma(B)\big)}$[2][3] *for all* $\lambda \in \rho(B)$.

[1]An operator valued map $B(\cdot)$ is called **analytic**, if it can be developed locally in a power series, i.e. if for every λ_0 in the domain there exist an $\epsilon > 0$ and a sequence (B_k) of bounded operators, such that $B(\lambda) = \sum_{k=1}^{\infty} (\lambda - \lambda_0)^k B_k$ for all λ in the domain with $|\lambda - \lambda_0| < \epsilon$.

[2]For any metric space (M, d), $M' \subseteq M$ and $m \in M$ the distance of the point m to the set M' is defined as $\operatorname{dist}(m, M') := \inf_{m' \in M'} d(m, m')$

[3]It is wellknown that $\|R_A(\lambda)\| = 1/\operatorname{dist}\big(\lambda, \sigma(A)\big)$ whenever X is a Hilbert space and A is a **normal operator**, i.e. $A^* A = A A^*$ (see Davies [4] p.247).

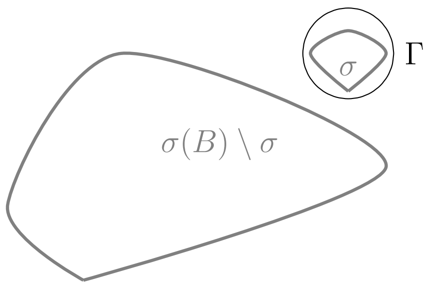

Figure 1.1: Γ separates σ from $\sigma(B) \setminus \sigma$.

Let σ be an isolated part of $\sigma(B)$, i.e. both σ and $\sigma(B) \setminus \sigma$ has to be closed, and Γ a piecewise continuously differentiable, counterclockwise oriented path with the property that σ belongs to its inner domain and $\sigma(B) \setminus \sigma$ to its outer domain (for illustration have a look at Figure 1.1).
Since the resolvent is analytic on $\rho(B)$, the **Riesz projection**

$$P_B(\sigma) := \frac{1}{2\pi i} \int_\Gamma R_B(\mu) d\mu$$

does not depend on the choice of Γ whenever σ is the only part of the spectrum which lies in the inner domain of Γ.
For the Riesz projection the following properties are used.

Proposition 1.4 *Let B be a bounded operator and σ be an isolated part of the spectrum.*

(i) $P_B(\sigma)^2 = P_B(\sigma)$,

(ii) $\mathrm{Ran}(P_B(\sigma))$ *and* $\mathrm{Ker}(P_B(\sigma))$ *are B-invariant*

(iii) $\sigma(B|_{\mathrm{Ran}(P_B(\sigma))}) = \sigma$ *and* $\sigma(B|_{\mathrm{Ker}(P_B(\sigma))})) = \sigma(B) \setminus \sigma$.

Proof: For a detaild proof we refer to Gohberg, Goldberg and Kasshoek [14] p. 326. □

Now, for any bounded operator B the **discrete spectrum** of B is

$$\sigma_d(B) := \{\lambda \in \sigma(B) : \lambda \text{ is isolated in } \sigma(B) \text{ and } \mathrm{Rank}\,(P_B(\{\lambda_0\})) < \infty\}.$$

For every $\lambda_0 \in \sigma(B)$ the integer $m_B(\lambda_0) := \mathrm{Rank}(P_B(\{\lambda_0\}))$ is the **algebraic multiplicity** of λ_0. Clearly, the discrete spectrum is at most countable. The set

$$\sigma_e(B) := \{\lambda \in \mathbb{C} \,:\, \lambda \mathbb{1} - B \text{ is not a Fredholm operator}\}$$

is called the **essential spectrum**, where an operator is called **Fredholm operator** iff both its kernel and cokernel are finite dimensional.
The dimension of the kernel of B is denoted by $n(B)$, the dimension of the cokernel is denoted by $d(B)$ and the integer

$$\mathrm{ind}(B) := n(B) - d(B).$$

is called the **index** of B.

Remark 1.5 Assume that $D \subseteq \mathbb{C}$ is open and $D \ni \lambda \mapsto B(\lambda)$ is continuous with respect to the operator norm, then

$$\lambda \mapsto \mathrm{ind}\,(B(\lambda))$$

is continuous (see [14] Theorem XI.4.1.). Moreover, since the expression $\mathrm{ind}(\cdot)$ is an integer, the function $\mathrm{ind}\,(B(\cdot))$ is locally constant.

Note that by definition it follows $\sigma_e(B) \subseteq \sigma(B)$ and that the essential spectrum is a closed set.
If the spectrum of B is already known it is possible to assign special points of $\sigma(B)$ to the essential spectrum, without using the definition of $\sigma_e(B)$.

Proposition 1.6 *Let B be a bounded operator and σ the non-discrete part of $\sigma(B)$, then*

$$\partial\sigma \subseteq \sigma_e(B).$$

Proof: Assume that this assertion is not true, i.e. there is a $\lambda_0 \in \partial\sigma$ which does not belong to the essential spectrum. Since the essential spectrum is a closed set, there has to be an $\epsilon > 0$ such that

$$\sigma_e(B) \cap \{\lambda \in \mathbb{C} \,:\, |\lambda - \lambda_0| < \epsilon\} = \emptyset.$$

Then the map

$$\{\lambda \in \mathbb{C} \,:\, |\lambda - \lambda_0| < \epsilon\} \ni \lambda \mapsto \lambda - B$$

is analytic and Fredholm valued.

Therefore the set

$$\Gamma := \{\mu \in \{\lambda \in \mathbb{C} \,:\, |\lambda - \lambda_0| < \epsilon\} \,:\, \mu - B \text{ is not invertible}\}$$

is, due to [14] Theorem XI.8.4, at most countable and has no accumulation points inside $\{\lambda \in \mathbb{C} \,:\, |\lambda - \lambda_0| < \epsilon\}$. But this is a contradiction to $\lambda_0 \in \partial\sigma$

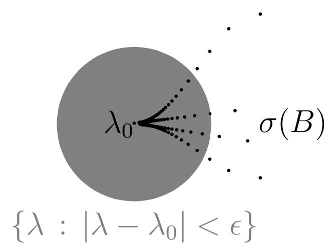

Figure 1.2: In this graphic, λ_0 is an element of $\partial\sigma$. As one can see, for every $\epsilon > 0$ there are points of $\rho(B)$ in $\{\lambda : |\lambda - \lambda_0| < \epsilon\}$, but also spectral points which do accumulate in $\{\lambda : |\lambda - \lambda_0| < \epsilon\}$.

(see Figure 1.2). $\square$

However, although the essential and the discrete spectrum are both parts of the spectrum there is no overlap of these two sets.

Proposition 1.7 *If $B \in \mathcal{L}(X)$ and λ is an isolated point of $\sigma(B)$, then $\lambda \in \sigma_{ess}(B)$ if and only if* $\mathrm{Rank}\,(P_B(\{\lambda\})) = \infty$ *and hence*

$$\sigma_d(B) \cap \sigma_e(B) = \emptyset. \tag{1.2}$$

Proof: The first assertion is proved in Davies [4] p. 122. To see (1.2) one can use the first assertion and the fact that a non-discrete part of the spectrum can not belong to the discrete spectrum by definition. $\square$

So, studying the discrete spectrum of a bounded operator means to study some set in the complement of the essential spectrum. To reduce the family of sets which are in the complement of the essential spectrum the next proposition characterizes the more relevant sets where one can find discrete eigenvalues (see [14] p. 373).

Proposition 1.8 *Let B be a bounded operator and $\Omega \nsubseteq \sigma_e(B)$ open with $\Omega \cap \rho(B) \neq \emptyset$ then $\Omega \cap \sigma(B) \subseteq \sigma_d(B)$.*

As a consequence

$$\sigma_d(B) \cup \sigma_e(B) \subseteq \sigma(B) \tag{1.3}$$

is an equality whenever B is an operator with connected set $\mathbb{C} \setminus \sigma_e(B)$.

Remark 1.9 Note, if B is an operator which has the property that its spectrum can be divided into discrete and essential part, then the only possible accumulation points of the discrete spectrum are in the essential spectrum.

Remark 1.10 In general there is no equality in (1.3) which can be easily seen in the following case:
Let $X := l^2(\mathbb{N})$ and S be the shift operator i.e. for every $f \in l^2(\mathbb{N})$ this operator is defined by $(Sf)(n) := f(n+1)$. Then $\sigma(S) = \overline{\mathbb{D}} := \overline{\{z \in \mathbb{C} \,:\, |z| < 1\}} = \{z \in \mathbb{C} \,:\, |z| \leq 1\}$, $\sigma_d(S) = \emptyset$ and $\sigma_e(S) = \partial\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| = 1\}$ (see also Chapter 6 or Section 7.3).

To conclude this section we need two perturbation results on the spectrum of bounded operators.

Theorem 1.11 *Let A be a bounded operator and $f : \Omega \to \mathbb{C}$ a holomorphic function with $\sigma(A) \subseteq \Omega$. Then*

$$\sigma(f(A)) = f(\sigma(A)) := \{f(\lambda) \,:\, \lambda \in \sigma(A)\}.$$

This theorem is known as spectral mapping theorem (see e.g. [14] p. 16). To clarify the expression $f(A)$ note that every holomorphic function f on some region of the complex plain which contains the spectrum of A can be developed locally into a powerseries $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with $(a_n) \subseteq \mathbb{C}$. Then $f(A) := \sum_{n=0}^{\infty} a_n(A - z_0\mathbb{1})^n$. For a more detailed introduction into holomorphic functional calculus have a look to [14] Part I.1.3.
So, we have seen that the spectrum depends holomorphically on holomorphic perturbations. The next theorem shows that also the discrete eigenvalues do have a continuous behaviour .

Theorem 1.12 ([14] p.33) *Let A be a bounded operator, σ be a finite set of eigenvalues of A and Γ some contour around σ which separates it from the rest of $\sigma(A)$. Then there exists an $\epsilon > 0$ such that for all operators B with $\|A - B\| < \epsilon$ the following holds true:*

$$\sigma(B) \cap \Gamma = \emptyset \text{ and}$$
$$\sum_{\lambda \text{ inside } \Gamma} m_B(\lambda) = \sum_{\lambda \text{ inside } \Gamma} m_A(\lambda).$$

1.2 Compact Operators

A bounded operator K is called a **compact operator** if for every bounded sequence (x_n), the sequence (Kx_n) has a convergent subsequence in X. We will denote the set of all compact operators on X by $\mathcal{S}_\infty(X)$. An operator F with $\mathrm{Rank}(F) < \infty$ is called **finite rank operator**, and of course there are the inclusions

$$\mathcal{F}(X) \subseteq \mathcal{S}_\infty(X) \subseteq \mathcal{L}(X). \tag{1.4}$$

Compact operators play a special role in spectral theory. For instance the index and consequently the essential spectrum are invariant under compact perturbations:

Theorem 1.13 ([14] p. Theorem XI. 4.1.) *Let B be a bounded operator and K a compact operator, then*

$$\mathrm{ind}(B) = \mathrm{ind}(B + K)$$

and

$$\sigma_e(B) = \sigma_e(B + K).$$

This theorem is known as Weyl's theorem.
In general, for bounded operators there is no rule for the distribution of the spectrum. However, for compact operators we have:

Theorem 1.14 (see e.g. [14] p. 30) *Let K be a compact operator on X. Then all non-zero elements of $\sigma(K)$ are discrete eigenvalues which can only accumulate at 0. If X is a infinite dimensional Banach space then $0 \in \sigma_e(K)$.*

It is not hard to see that the set of compact operators forms an **ideal**, i.e. for any operator $B \in \mathcal{L}(X)$ and $K \in \mathcal{S}_\infty(X)$

$$BK \in \mathcal{S}_\infty(X) \text{ and } KB \in \mathcal{S}_\infty(X).$$

One needs more effort to show that $S_\infty(X)$ is closed with respect to the operator norm, therefore we refer to [4] p. 103 for a detailed proof.

According to (1.4) every finite rank operator is a compact operator. To illustrate the quantity of the distance of a compact operator to the set of finite rank operators we introduce the **approximation numbers**. So, for $K \in \mathcal{S}_\infty(X)$ and $n \in \mathbb{N}$

$$\alpha_n(K) := \inf\{\|K - F\| \, : \, \mathrm{Rank}(F) < n\}$$

defines the **nth approximation number**.
Although this section is devoted to compact operators, the definition of approximation numbers runs also if we replace compact by bounded operators, and in this case we have:

Proposition 1.15 (see e.g. Pietsch [30] Sections 2.2 and 2.3) *Let $A, B, C \in \mathcal{L}(X)$, then:*

(i) $\|A\| = \alpha_1(A) \geq \alpha_2(A) \geq \cdots \geq 0$,

(ii) $\alpha_{n+m+1}(A+B) \leq \alpha_n(A) + \alpha_m(B)$,

(iii) $\alpha_n(ABC) \leq \|A\|\alpha_n(B)\|C\|$,

(iv) *if* $\mathrm{Rank}(A) < n$, *then* $\alpha_n(A) = 0$.

There is also an interesting and important connection between the eigenvalues of a compact operator and its approximation numbers.

Theorem 1.16 (see König [26] Theorem 2.a.6) *Let $p \in (0, \infty)$, then for any compact operator $K \in \mathcal{S}_\infty(X)$ and any $n \in \mathbb{N}$ we have*

$$\sum_{i=1}^{n} |\lambda_i(K)|^p \leq 2(2e)^{\frac{p}{2}} \sum_{i=1}^{n} \alpha_i(K)^p, \tag{1.5}$$

where $(\lambda_i(K))$ denotes the sequence of discrete eigenvalues of K counted with their algebraic multiplicity.

(1.5) is the generalized Weyl's inequality for Banach spaces, where the classical Weyl's inequality [35] is formulated for compact operators K in Hilbert spaces and states for any $p \in (0, \infty)$ and $n \in \mathbb{N}$

$$\sum_{i=1}^{n} |\lambda_i(K)|^p \leq \sum_{i=1}^{n} \alpha_i(K)^p.$$

As we will see, the summability of the eigenvalues of a compact operator will play an important role in the following.
It turns out that certain Banach ideals, or quasi-Banach ideals will play an essential role.

Chapter 2

Certain quasi-Banach and Banach ideals

The term **quasi-Banach ideal** denotes a tuple $\big(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}}, c_{\mathcal{M}}\big)$, where $\mathcal{M}(X) \subseteq \mathcal{L}(X)$ is an ideal and $\|\cdot\|_{\mathcal{M}} : \mathcal{M}(X) \to \mathbb{R}_+$ is a positive map which satisfies all conditions of a norm except the triangle inequality, i.e. instead of this inequality we have that there is a $c_{\mathcal{M}} \geq 1$ such that

$$\|A+B\|_{\mathcal{M}} \leq c_{\mathcal{M}}(\|A\|_{\mathcal{M}} + \|B\|_{\mathcal{M}}) \text{ for all } A, B \in \mathcal{M}(X).$$

If $\mathcal{M}(X)$ is complete with respect to $\|\cdot\|_{\mathcal{M}}$, then the tuple $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}}, c_{\mathcal{M}})$ is a quasi-Banach ideal.

If $c_{\mathcal{M}} = 1$, then in fact $\|\cdot\|_{\mathcal{M}}$ is a norm, and in this case we call the pair $\big(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}}\big)$ a **Banach ideal**.

For reasons of simplicity we will write for a Banach or quasi-Banach ideal only $\mathcal{M}(X)$ instead of $\big(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}}, c_{\mathcal{M}}\big)$ or $\big(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}}\big)$ if there is no question of confusion.
In the Hilbert space theory the pth Schatten class denotes the Banach ideal of all compact operators with p-summing approximation numbers where the underlying ideal norm is denoted by the l^p-norm of the approximation numbers. It seems to be natural to extend the term Schatten class in the same way to general Banach spaces. The problem of this procedure is, that this set fails to be a Banach ideal.
However, there is a rich theory for extending the Schatten class operators to Banach spaces (see e.g. Pietsch [30]).

2.1 The Banach ideal of nuclear operators

Definition 2.1 Let K be a compact operator in $\mathcal{L}(X)$ (the space of linear bounded operators). K is called **nuclear** if there are sequences (not necessarily unique) $\{f_n\} \subseteq X$, $\{\phi_n\} \subseteq X^*$ (the dual of X) such that Kf can be represented by

$$Kf = \sum_{n=1}^{\infty} \langle \phi_n, f \rangle f_n$$

for all $f \in X$ and

$$\sum_{n=1}^{\infty} \|\phi_n\|_{X^*} \|f_n\|_X < \infty.$$

We denote this class by $\mathcal{N}(X)$

In $\mathcal{N}(X)$ a norm can be defined by

$$\|K\|_{\mathcal{N}} := \inf\{\sum_{n=1}^{\infty} \|\phi_n\|_{X^*} \|f_n\|_X \ : \ Kf = \sum_{n=1}^{\infty} \langle \phi_n, f \rangle f_n \text{ for all } f \in X\}.$$

With this norm $\mathcal{N}(X)$ becomes a Banach ideal (see Pietsch [30], Section 1.7).

Examples 2.2 (a) Let (e_k) be the canonical standard basis in $l^p(\mathbb{Z})$ with $1 \le p \le \infty$. Denote by ϕ_m the sequence $\phi_m = (a_{mj})_{j \in \mathbb{Z}} \in l^q(\mathbb{Z})$ $(\frac{1}{p} + \frac{1}{q} = 1)$. Assuming $(\|\phi_m\|_q)_{m \in \mathbb{Z}} \in l^1(\mathbb{Z})$, then the operator $K : l^q(\mathbb{Z}) \to l^q(\mathbb{Z})$ defined by $Kf := \sum_{m \in \mathbb{Z}} \langle \phi_m, f \rangle e_m$ is nuclear. The corresponding infinite diagonal matrix is given by $(a_{mj})_{m,j \in \mathbb{Z}}$.
We can conclude that every diagonal operator which is defined by an infinite matrix diag$(..., d_{-1}, d_0, d_1, ...)$ is nuclear if $\{d_n\}_{n \in \mathbb{Z}} \in l^1(\mathbb{Z})$.
If K acts on the space $l^1(\mathbb{Z})$ then the nuclear norm is given by (see Gohberg, Goldberg and Krupnik [15], Chapter 2 Theorem 2.1)

$$\|K\|_{\mathcal{N}} = \sum_{m=-\infty}^{\infty} \sup_{j \in \mathbb{Z}} |a_{mj}|$$

(b) Every integral operator

$$K : C([\alpha, \beta]) \to C([\alpha, \beta]), \ (Kf)(t) := \int_{\alpha}^{\beta} k(t,s) f(s) ds$$

with continuous kernel k is nuclear and $\|K\|_{\mathcal{N}} \leq \int_\alpha^\beta \sup_{t\in[\alpha,\beta]} |k(t,s)|ds$. For each $n \in \mathbb{N}$ let $R_n := (s_{nk})_{k=1}^{N_n}$ be a partition of the interval $[a,b]$ such that

$$a = s_{n0} < s_{n1} < \cdots < s_{nN_n} = b$$
$$R_m \supset R_n \text{ for all } m > n \text{ and}$$
$$s_{n,k+1} - s_{nk} \overset{n\to\infty}{\to} 0 \text{ for all } k.$$

Then

$$K_n f := \sum_{p=0}^{N_n} \langle \phi_{pn}, f\rangle f_{pn} \quad f \in C[\alpha,\beta],$$

where $f_{pn}(t) := k(t, s_{np})$ and $\phi_{pn}(g) := \int_{s_{np}}^{s_{n,p+1}} g(s)ds$, defines a nuclear operator with (see [15], Chapter 2 Theorem 2.2)

$$\|K_n - K\|_{\mathcal{N}} \overset{n\to\infty}{\to} 0.$$

Hence, K is nuclear with

$$\begin{aligned}\|K\|_{\mathcal{N}} = \lim_{n\to\infty} \|K_n\|_{\mathcal{N}} &\leq \sum_{p=1}^{N_n} \|\phi_{pn}\|_{C[\alpha,\beta]^*} \|f_{pn}\|_{C[\alpha,\beta]} \\ &\leq \lim_{n\to\infty} \sum_{p=1}^{N_n} \sup_{t\in[\alpha,\beta]} |k(t,s_{np})|(s_{n,p+1} - s_{np}) = \int_\alpha^\beta \sup_{t\in[\alpha,\beta]} |k(t,s)|ds.\end{aligned}$$

Remark 2.3 If X is a Hilbert space $\mathcal{N}(X)$ coincides with the ideal of trace class operators. In this case we know that the eigenvalues are summable. However, there are Banach spaces and nuclear operators with non summable eigenvalues (see e.g. Gohberg, Goldberg and Krupnik [15] p. 102).

Nevertheless, in general one has the following estimate:

Theorem 2.4 *Let $\{\lambda_n(K)\}$ be the eigenvalues of the nuclear operator K, then*

$$\sum_{n=1}^{\infty} |\lambda_n(K)|^2 \leq \|K\|_{\mathcal{N}}^2, \tag{2.1}$$

(see e.g. Pietsch [30], p. 160).

Example 2.5 If X_1 and X_2 are compatible[1] Banach spaces and if K_1 and K_2 are consistent[2] compact operators acting in X_1 and X_2 then (see [4] p. 109).

$$\sigma(K_1) = \sigma(K_2).$$

We know that for $1 \leq p_1, p_2 < \infty$ the spaces $l^{p_1}(\mathbb{N})$ and $l^{p_2}(\mathbb{N})$ are compatible. Now let K_1 be an operator on $l^1(\mathbb{N})$ and K_2 be an operator on $l^2(\mathbb{N})$ and let K_1 and K_2 be consistent. If the eigenvalues of K_1 are square summable the same is true for K_2. Now let K_2 be an operator defined on $l^2(\mathbb{N})$ which is consistent to a nuclear operator K_1 defined on $l^1(\mathbb{N})$. Then K_2 is not automatically a Hilbert-Schmidt operator or a trace-class operator.
To check this we define the infinite matrix

$$(a_{km})_{k,m\in\mathbb{N}} := \begin{pmatrix} 2^{-1} & 2^{-1} & 2^{-1} & \dots \\ 2^{-2} & 2^{-2} & 2^{-2} & \dots \\ 2^{-3} & 2^{-3} & 2^{-3} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and define with this matrix the operators K_1 and K_2.
For K_1 the nuclear norm is $\|K_1\|_{\mathcal{N}} = \sum_{k=1}^{\infty} \sup_m |a_{km}|$ (see [15], Chapter V Theorem 2.1). So we have

$$\|K_1\|_{\mathcal{N}} = \sum_{k=1}^{\infty} 2^{-k} = 1$$

such that K_1 is in fact a nuclear operator.
K_2 is a Hilbert-Schmidt operator on $l^2(\mathbb{N})$ iff the sum $\sum_{j=1}^{\infty} \|K_2 e_j\|_2$ is finite, where (e_j) is the orthonormal standard basis in $l^2(\mathbb{N})$ (see [15], Chapter IV Theorem 7.1).In the present example

$$\sum_{j=1}^{\infty} \|K_2 e_j\|_2^2 = \sum_{j=1}^{\infty} \|(2^{-k})\|_2^2 = \infty,$$

that means K_2 is not a Hilbert-Schmidt operator and hence not a trace class operator.

[1]Two Banach spaces X_1 and X_2 are called **compatible** if $X_1 \cap X_2$ is dense in X_1 and X_2.

[2]The operators K_1 and K_2 are called **consistent**, if they coincide on $X_1 \cap X_2$.

2.2 The quasi-Banach ideal of operators of type l^p

As mentioned in the beginning of this chapter, if X is a Hilbert space, the family $\mathcal{S}_p(X) := \{B \in \mathcal{L}(X) : \big(\alpha_n(B)\big) \in l^p(\mathbb{N})\}$ of p**th Schatten class operators** together with the norm $\|B\|_{\mathcal{S}_p} = \big(\sum_{n=1}^{\infty} \alpha_n(B)^p\big)^{\frac{1}{p}}$ defines a Banach ideal for all $p \geq 1$. If X is a Banach space we call $\mathcal{S}_p(X)$ **operators of type** l^p. In general (X a Banach space) $\mathcal{S}_p(X)$ fails to be a Banach-ideal for all $p > 0$. In fact, if $A, B \in \mathcal{S}_p(X)$ we have due to Proposition 1.15 (ii)

$$\alpha_{2n+1}(A+B) \leq \alpha_{n+1}(A) + \alpha_{n+1}(B), \tag{2.2}$$
$$\alpha_{2n+2}(A+B) \leq \alpha_{n+1}(A) + \alpha_{n+2}(B). \tag{2.3}$$

If $0 < p < 1$ we can use (2.2), (2.3) and $(a+b)^p \leq a^p + b^p$ $(a, b \geq 0)$ to obtain

$$\sum_{n=0}^{\infty} \alpha_{2n+1}(A+B)^p \leq \sum_{n=0}^{\infty} \alpha_{n+1}(A)^p + \sum_{n=0}^{\infty} \alpha_{n+1}(B)^p \leq \|A\|_{\mathcal{S}_p}^p + \|B\|_{\mathcal{S}_p}^p \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} \alpha_{2n+2}(A+B)^p \leq \sum_{n=0}^{\infty} \alpha_{n+1}(A)^p + \sum_{n=0}^{\infty} \alpha_{n+2}(B)^p \leq \|A\|_{\mathcal{S}_p}^p + \|B\|_{\mathcal{S}_p}^p. \tag{2.5}$$

(2.4) and (2.5) together imply

$$\|A+B\|_{\mathcal{S}_p}^p \leq 2\left(\|A\|_{\mathcal{S}_p}^p + \|B\|_{\mathcal{S}_p}^p\right)$$

and therefore, using the inequality $(a+b)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}\left(a^{\frac{1}{p}} + b^{\frac{1}{p}}\right)$ $(a, b \geq 0)$,

$$\|A+B\|_{\mathcal{S}_p} \leq \left(2\left(\|A\|_{\mathcal{S}_p}^p + \|B\|_{\mathcal{S}_p}^p\right)\right)^{\frac{1}{p}} \leq 2^{\frac{2}{p}-1}\left(\|A\|_{\mathcal{S}_p} + \|B\|_{\mathcal{S}_p}\right).$$

For $p \geq 1$, (2.2), (2.3) and the Minkowski inequality give

$$\begin{aligned}
\big(\sum_{n=0}^{\infty} \alpha_{2n+1}(A+B)^p\big)^{1/p} &\leq \big(\sum_{n=0}^{\infty} \alpha_{n+1}(A)^p\big)^{1/p} + \big(\sum_{n=0}^{\infty} \alpha_{n+1}(B)^p\big)^{1/p} \\
&\leq \|A\|_{\mathcal{S}_p} + \|B\|_{\mathcal{S}_p}, \\
\big(\sum_{n=0}^{\infty} \alpha_{2n+2}(A+B)^p\big)^{1/p} &\leq \big(\sum_{n=0}^{\infty} \alpha_{n+1}(A)^p\big)^{1/p} + \big(\sum_{n=0}^{\infty} \alpha_{n+2}(B)^p\big)^{1/p} \\
&\leq \|A\|_{\mathcal{S}_p} + \|B\|_{\mathcal{S}_p},
\end{aligned}$$

and hence,

$$\|A+B\|_{\mathcal{S}_p} \leq 2(\|A\|_{\mathcal{S}_p} + \|B\|_{\mathcal{S}_p}).$$

All in all, the inequality

$$\|A+B\|_{\mathcal{S}_p} \leq c_{\mathcal{S}_p} \left(\|A\|_{\mathcal{S}_p} + \|B\|_{\mathcal{S}_p}\right)$$

holds for all $A, B \in \mathcal{S}_p(X)$ with

$$c_{\mathcal{S}_p} = \begin{cases} 2^{\frac{2}{p}-1}, 0 < p < 1, \\ 2, p \geq 1. \end{cases}$$

In order to see that the constant $c_{\mathcal{S}_p}$ is equal to 1 if X is a Hilbert space and $p \geq 1$, we can use the formula

$$\|A\|_{\mathcal{S}_p} = \sup_{F \in \mathcal{F}(X)} \left| \frac{\operatorname{tr}(AF)}{\|F\|_{\mathcal{S}_q}} \right|,$$

where $q^{-1} + p^{-1} = 1$ and $\operatorname{tr}(F) := \sum_{\lambda \in \sigma(F)} \lambda$ defines the (linear) **trace** of F (see e.g. [9] p. 1098).
As a direct consequence of Theorem 1.16 there is the following connection between the eigenvalue sequence of any operator of type l^p and the quasi-norm $\|\cdot\|_{\mathcal{S}_p}$.

Theorem 2.6 *If $A \in \mathcal{S}_p(X)$ then*

$$\sum_{n=1}^{\infty} |\lambda_n(A)|^p \leq c_p \|A\|_{\mathcal{S}_p}^p$$

with $C_p = 2(2e)^{\frac{p}{2}}$.

Remark 2.7 The eigenvalue inequality in Theorem 2.6 gives not only the rate of accumulation of the discrete eigenvalues to the point 0, but also information on the bound of the number of discrete eigenvalues in the complement of some disc, $(r\mathbb{D})^c$, that is

$$c_p \|A\|_{\mathcal{S}_p}^p \geq \sum_{\lambda \in \sigma(A)} |\lambda|^p \geq \sum_{\lambda \in \sigma(A), |\lambda| \geq r} |\lambda|^p \geq \#\left(\sigma(A) \cap (r\mathbb{D})^c\right) r^p$$

and therefore

$$\#\left(\sigma(A) \cap (r\mathbb{D})^c\right) \leq \frac{c_p \|A\|_{\mathcal{S}_p}^p}{r^p}.$$

2.3 The Banach ideal of p-summing operators

One other generalization of Schatten class operators is the space $\Pi_p(X,Y)$ of **p-summing operators** ($p \geq 1$). We call a bounded operator $A \in \mathcal{L}(X,Y)$[3] p-summing, if there is a constant $c \geq 0$ such that

$$\Big(\sum_{i=1}^{m} \|Ax_i\|_Y^p\Big)^{\frac{1}{p}} \leq c \sup\Big\{\Big(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p\Big)^{\frac{1}{p}} \;:\; \|x^*\|_{X^*} \leq 1\Big\} \tag{2.6}$$

for all $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$ and denote the space of all such operators with $\Pi_p(X,Y)$ (if $X = Y$ then $\Pi_p(X,X) = \Pi_p(X)$). The map $\|\cdot\|_{\Pi_p} : \Pi_p(X,Y) \to \mathbb{R}_+$ with $\|A\|_{\Pi_p} := \inf\{c\,;\, c \text{ satisfies } (2.6)\}$ defines a norm. Since the proof of this assertion is very simple we will only state the proof of the triangle inequality. Let us assume that there are operators $A, B \in \Pi_p(X,Y)$ with $\|A\|_{\Pi_p} + \|B\|_{\Pi_p} < \|A+B\|_{\Pi_p}$. Then one can choose $c_1 > \|A\|_{\Pi_p}$ and $c_2 > \|B\|_{\Pi_p}$ such that

$$c := c_1 + c_2 < \|A+B\|_{\Pi_p},$$

which implies

$$\begin{aligned}\Big(\sum_{i=1}^{m} \|(A+B)x_i\|_Y^p\Big)^{\frac{1}{p}} \leq& \Big(\sum_{i=1}^{m} \|Ax_i\|_Y\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{m} \|Bx_i\|_Y\Big)^{\frac{1}{p}} \\ \leq& c \sup\Big\{\Big(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p\Big)^{\frac{1}{p}} \;:\; \|x^*\|_{X^*} \leq 1\Big\}\end{aligned}$$

for all $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$. But in this case $\|A+B\|_{\Pi_p}$ fails to be the infimum over all numbers which satisfy (2.6) for $A+B$.
It is not hard to see that $\Pi_p(X,Y)$ defines a Banach ideal using the definition, but there is also another skilful approach which one can find in Diestel, Jarchow, Tonge [8] p. 37.
Similar to the other examples in this subsection also the Banach ideal $\Pi_p(X)$ provides an eigenvalue estimate in terms of the p-summing norm.

Theorem 2.8 *Let $1 \leq p < \infty$ and $A \in \Pi_p(X)$, then*

$$\sum_{i=1}^{\infty} |\lambda_i(A)|^r \leq \|A\|_{\Pi_r}^r, \tag{2.7}$$

with $r := \max\{2, p\}$.

[3] Let X and Y be normed spaces. Then the family of all bounded operators $A : X \to Y$ is denoted by $\mathcal{L}(X,Y)$.

Proof: One has to prove the cases $1 \leq p \leq 2$ and $p > 2$ separately. The case for $p > 2$ was done by Johnson, König, Maurey and Retherford in 1979 [24] and is harder than the other one. So, we will only sketch the proof for $1 \leq p \leq 2$, which is due to Pietsch [30] p. 158.
Since $\Pi_p(X) \subseteq \Pi_q(X)$ whenver $p \leq q$ it is enough to assume $A \in \Pi_2$. It is known that there exists a Hilbert space H and operators $B \in \Pi_2(X,H)$, $C \in \mathcal{L}(H,X)$ such that $A = BC$ and $\|C\|\|B\|_{\Pi_2} = \|A\|_{\Pi_2}$ (see [30] Factorization Theorem p. 56). Then $CB \in \Pi_2(H) = S_2(H)$ is a Hilbert-Schmidt operator and we have $\sigma(CB) = \sigma(BC) = \sigma(A)$. At least we can derive

$$\big(\sum_{n=1}^{\infty} |\lambda_n(BC)|^2\big)^{\frac{1}{2}} = \big(\sum_{n=1}^{\infty} |\lambda_n(CB)|^2\big)^{\frac{1}{2}} \leq \|CB\|_{\Pi_2} \leq \|C\|\|B\|_{\Pi_2} = \|A\|_{\Pi_2}.$$

$\square$

Remark 2.9 Pietsch showed, that in general, $r = 2$ is the optimal exponent for the case $1 \leq p \leq 2$ ([30] Section 3.7.1).

Remark 2.10 If $A \in \mathcal{N}(X)$ is a nuclear operator, then for every $\epsilon > 0$ there has to be a sequence $(f_i) \subseteq X$ and a seuqence $(\phi_i) \subseteq X^*$ with

$$Af = \sum_{i=1}^{\infty} \langle \phi_i, f\rangle f_i \text{ for } f \in X \text{ and } \sum_{i=1}^{\infty} \|\phi_i\|_{X^*}\|f_i\|_X \leq \|A\|_{\mathcal{N}} + \epsilon.$$

therefore for every finite system $x_1, \ldots, x_n \in X$ one gets

$$\begin{aligned}\sum_{i=1}^{n} \|Ax_i\|_X &\leq \sum_{i=1}^{n}\sum_{j=1}^{\infty} \|\phi_j\|_{X^*} \left|\left\langle \frac{\phi_j}{\|\phi_j\|_{X^*}}, x_i\right\rangle\right| \|f_j\|_X \\ &\leq \sum_{j=1}^{\infty} \|\phi_j\|_{X^*}\|f_j\|_X \sum_{i=1}^{n} \left|\left\langle \frac{\phi_j}{\|\phi_j\|_{X^*}}, x_i\right\rangle\right| \leq \big(\|A\|_{\mathcal{N}} + \epsilon\big) \sup\{|\langle x, x_i\rangle|, x \in X^*\},\end{aligned} \tag{2.8}$$

hence $\mathcal{N}(X)$ the space of nuclear operators is contained in $\Pi_p(X)$ for all $p \geq 1$.
Moreover, (2.8) shows that

$$\|A\|_{\Pi_p} \leq \|A\|_{\mathcal{N}} \text{ for all } A \in \mathcal{N}(X)$$

and therefore $\mathcal{N}(X) \subseteq \Pi_p^{\mathcal{F}}(X)$. Here $\Pi_p^{\mathcal{F}}(X)$ denotes the topological closure of $\mathcal{F}(X)$ with respect to $\|\cdot\|_{\Pi_p}$.

At least one should note, that all preceding ideals in this section are subsets of the ideal of compact operators. But that is not true for $p-$summing operators. In general a $p-$summing operator is only **weakly compact** (see e.g. [8] p. 50). I.e. if $(x_n) \subseteq X$ is $\|\cdot\|_X$-bounded, then $(\langle x, Ax_n\rangle)$ has a convergent subsequence for every $x \in X^*$.

2.4 Ideal of compact operators of infinite order

In the preceding sections operators with p-summing eigenvalues were introduced, in this section we only assume that the approximation numbers satisfy

$$\lim_{n\to\infty} \alpha_n(K) = 0. \tag{2.9}$$

Remark 2.11 Note, that every compact operator on a Hilbert space fulfills (2.9), i.e. is the limit of finite rank operators.
Indeed, if $A \in \mathcal{S}_\infty(X)$ then the approximation numbers $(\alpha_n(A))$ coincide with the **singular values** $(s_n(A))$ (see e.g. [14] p.98) which is defined to be the monotone decreasing sequence $(\sqrt{\lambda_n(A^*A)})$, where $(\lambda_n(A^*A))$ is the positive sequence of the repeated eigenvalues (with respect to the multiplicity) of the compact[4] operator A^*A. Hence, since A^*A is compact its eigenvalue sequence tends to zero and also so the approximation numbers (singular values).
In general Banach spaces that is not the case. In fact, Enflo [10] constructed a Banach space on which there are compact operators which are not the operator norm limits of finite rank operators.
If a Banach space has the property that every compact operator is the limit of finite rank operators, then we say that this space has the **approximation property**.

Under assumption (2.9) it is possible to define a new type of ideals:
Given a sequence of integers $(p_n) \subseteq \mathbb{N}$ we say a compact operator A is of **type** $l^{(p_n)}$ if

$$\sum_{n=1}^{\infty} (t\alpha_n(A))^{p_n} < \infty \text{ for all } t \geq 0.$$

We will denote this set of operators by $\mathcal{S}_{(p_n)}(X)$. Using Proposition 1.15 a short calculation shows that this family denotes an ideal:

$$\sum_{n=1}^{\infty} (t\alpha_n(BAC))^{p_n} \leq \sum_{n=1}^{\infty} (t\|B\|\alpha_n(A)\|C\|)^{p_n} < \infty.$$

[4] If A is compact the adjoint A^* has also to be compact and therefore also the product.

If an operator $A \in \mathcal{S}_{(p_n)}(X)$ then also $A \in \mathcal{S}_{(p_n+1)}(X)$. The moments of eigenvalues can be estimated.

Theorem 2.12 *If $A \in \mathcal{S}_{(p_n)}(X)$ then*

$$\sum_{n=1}^{\infty} |\lambda_n(A)|^{p_n} \leq \sum_{n=1}^{\infty} (c\alpha_n(A))^{p_n}$$

with $c = 2(2e)^{\frac{1}{2}}$.

Proof:Due to Theorem 1.16

$$\sum_{n=1}^{N} |\lambda_n(A)| \leq \sum_{n=1}^{N} c\alpha_n(K)$$

holds for every $N \in \mathbb{N}$ with $c = 2(2e)^{\frac{1}{2}}$.
According to a theorem by Marcus ([34] Theorem 1.9), the inequality

$$\phi\left(|\lambda_1(A)|, \ldots, |\lambda_N(A)|\right) \leq \phi\left(c\alpha_1(A), \ldots, c\alpha_N(A)\right)$$

is also true, if ϕ is a convex function.
Since

$$\phi(x_1, \ldots, x_N) := \sum_{n=1}^{N} x^{p_n}$$

is convex, the assertion is true. $\square$

Remark 2.13

- If we set $p_n \geq n$ then

$$\mathcal{S}_{(p_n)}(X) = \{A \in \mathcal{S}_{\infty}(X) : \lim_{n \to \infty} \alpha_n(A) = 0\}.$$

- If $p_n = p$ is constant, then

$$\mathcal{S}_{(p_n)}(X) = \mathcal{S}_p(X).$$

Chapter 3

Determinants for compact operators

It is well known, that the eigenvalues of a finite rank operator $A : \mathbb{C}^n \to \mathbb{C}^n$ are exactly the zeros of the characteristic polynomial

$$\chi_A(\lambda) := \det(\lambda - A), \tag{3.1}$$

where the **determinant** of an operator $B : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$\det(B) := \sum_{\pi \in \mathbb{P}_n} \big(\operatorname{sgn}(\pi) \prod_{i=1}^{n} b_{i,\pi(i)}\big). \tag{3.2}$$

Here $\mathbb{P}_n$ denotes the set of all permutations of $\{1, \dots, n\}$,

$$\operatorname{sgn}(\pi) := \frac{\prod_{1 \le i < j \le n} \big(\pi(j) - \pi(i)\big)}{\prod_{1 \le i < j \le n} (j - i)}$$

is the sign of the permutation π and $(b_{i,j})_{1 \le i,j \le n}$ is the matrix which defines the operator according to some basis of $\mathbb{C}^n$.
The term algebraic multiplicity introduced in Section 1 is also known to be the order of the zeros of the characteristic polynomial. For convenience of the reader the next proposition shows that these two definitions are consistent in finite dimensional spaces.

Proposition 3.1 *Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be an operator. $\lambda_0 \in \mathbb{C}$ is an eigenvalue of A with algebraic multiplicity m if and only if λ_0 is a zero of χ_A of order m.*

Proof: A is unitary equivalent to its Jordan normal form

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_d \end{pmatrix} \text{ with } J_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_j \end{pmatrix} \in \mathbb{C}^{m_j \times m_j}.$$

Hence $\chi_A(\lambda) = \chi_J(\lambda) = \prod_{j=1}^{d} (\lambda - \lambda_j)^{m_j}$ and for each $i = 1, \ldots, d$ the value λ_i is a zero of χ_A of order m_i.
To determine the rank of the Riesz projection of A according to λ_i it suffices to determine the rank fo the Riesz projection of J according to λ_i.
For this let Γ_i be some closed curve around λ_i separating λ_i from all the other eigenvalues of A. Then

$$\frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - J)^{-1} d\lambda = \begin{pmatrix} \tilde{J}_1 & & \\ & \ddots & \\ & & \tilde{J}_d \end{pmatrix} \text{ with}$$

$$\tilde{J}_j = \frac{1}{2\pi i} \begin{pmatrix} \int_{\Gamma_i} (\lambda - \lambda_j)^{-1} d\lambda & \ldots & \int_{\Gamma_i} (\lambda - \lambda_j)^{-m_j} d\lambda \\ & \ddots & \vdots \\ & & \int_{\Gamma_i} (\lambda - \lambda_j)^{-1} d\lambda \end{pmatrix}.$$

Moreover, $\tilde{J}_j$ is the identity-matrix in $\mathbb{C}^{m_j \times m_j}$ if $j = i$, and the zero-matrix else, and therefore

$$\text{Rank} \left(\frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - A)^{-1} d\lambda \right) = \text{Rank} \left(\frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - J)^{-1} d\lambda \right) = m_i.$$

□

However, if X is an infinite dimensional Banach space and A a bounded operator on X, there are problems to generalize (3.2), even if there is an infinite matrix-representation of A (e.g. what is sgn of an infinite permutation?).
Hence, we need another (equivalent) representation for the determinants.
As we have seen in the proof of Proposition 3.1 for any operator
$A : \mathbb{C}^n \to \mathbb{C}^n$ we have

$$\chi_A(\lambda) = \prod_{i=1}^{d} (\lambda - \lambda_i)^{m_i},$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of A with corresponding algebraic multiplicities $m_1, \dots, m_d$ ($\sum_{i=1}^d m_i = n$). Then

$$(-1)^n \chi_A(0) = \prod_{\lambda \in \sigma(A)} \lambda = \det(A).$$

So, the determinant is the product of all repeated eigenvalues, i.e. counted with their algebraic multiplicity.
Now let X be an infinite dimensional Banach space, let $A \in \mathcal{S}_\infty(X)$ be a compact operator, and $\big(\lambda_i(A)\big)$ its corresponding sequence of repeated eigenvalues, counted with their algebraic multiplicity. We define

$$\det{}_1(\mathbb{1} - A) := \prod_{i=1}^{\infty} \big(1 - \lambda_i(A)\big). \tag{3.3}$$

According to the theory of infinite products (see e.g. [1] p. 191-192), (3.3) converges if and only if

$$\sum_{i=1}^{\infty} |\lambda_i(A)| < \infty. \tag{3.4}$$

According to Theorem 2.6 (3.4) is true for all $A \in \mathcal{S}_1(X)$.
To extend the set of operators for which a useful[1] determinant can be defined one can add a regularizing factor:

$$\det{}_{\lceil p \rceil}(\mathbb{1} - A) := \sum_{j=1}^{\infty} \big(1 - \lambda_j(A)\big) \exp\left(\sum_{n=1}^{\lceil p \rceil - 1} \frac{\lambda_j(A)^n}{n} \right) \tag{3.5}$$

The term $\det_{\lceil p \rceil}$ is well-defined for all $A \in \mathcal{S}_p(X)$, $A \in \Pi_p(X)$ if $p \geq 2$ and $A \in \mathcal{N}(X)$ if $p = 2$, and is called $\lceil p \rceil$**th regularized determinant**. The fact, that $\det_{\lceil p \rceil}$ is well defined for p-summing operators, operators of type l^p or nuclear operators follows by the lemma below (see e.g. [9] p. 1107).

Lemma 3.2 *For any $p > 0$ there exists a constant $\Gamma_p > 0$ such that*

$$\left| (1 - z) \exp\left(\sum_{j=1}^{n-1} \frac{z^j}{j} \right) \right| \leq \exp\big(\Gamma_n |z|^n\big) \text{ for all } z \in \mathbb{C}.$$

Remark 3.3 For $p \leq 1$ we have $\Gamma_p \leq \frac{1}{p}$. For an integer $p \geq 2$ we have $\Gamma_p \leq \frac{p-1}{p}$ if $p \neq 3$ and $\Gamma_3 \leq 1$ (see e.g. [13], [28] p. 225).

[1] This term will be explained in the next section.

Using Lemma 3.2 one also obtains an upper bound in terms of the corresponding ideal norms.

Theorem 3.4 *1)* $|\det_2(\mathbb{1} - A)| \leq \exp\left(\frac{1}{2}\|A\|_{\mathcal{N}}^2\right)$ *for all* $A \in \mathcal{N}(X)$*,*

2) $|\det_{\lceil p\rceil}(\mathbb{1} - A)| \leq \exp\left(c_p\Gamma_p\|A\|_{\mathcal{S}_p}^p\right)$ *for all* $A \in \mathcal{S}_p(X)$ *for all* $p \geq 1$*,*

3) $|\det_{\lceil p\rceil}(\mathbb{1} - A)| \leq \exp\left(\Gamma_p\|A\|_{\Pi_p}^p\right)$ *for all* $A \in \Pi_p(X)$ *for all* $p \geq 2$ *and*

4) $|\det_2(\mathbb{1} - A)| \leq \exp\left(\frac{1}{2}\|A\|_{\Pi_2}^2\right)$ *for all* $A \in \Pi_p(X)$ *for all* $2 > p \geq 1$*,*
where Γ_p *is as in Lemma 3.2 and* $c_p = 2(2e)^{\frac{p}{2}}$*.*

Proof: Combine Lemma 3.2 with Theorem 2.4, 2.6, 2.8. As a demonstration for all other ideals we give the proof for the ideal of nuclear operators:

$$|\det_2(\mathbb{1} - A)| \leq \exp\left(\frac{1}{2}\sum_{i=1}^{\infty}|\lambda_i(A)|^2\right) \leq \exp\left(\frac{1}{2}\|A\|_{\mathcal{N}}^2\right) \text{ for all } A \in \mathcal{N}(X).$$

□

Lemma 3.2 shows that such a regularized determinant $\det_{\lceil p\rceil}$ is well defined for any operator with eigenvalue sequences belonging to $l^p(\mathbb{N})$. But this is not true for every compact operator. There are also compact operators the eigenvalue sequences and approximation numbers of which behave like $(\frac{1}{\log(n)})$ (e.g. a diagonal operator defined on $l^2(\mathbb{N})$ with $1/(\log(n)+1)$ on the main diagonal). In this case one has to change the index of summation in the regularizing factor of (3.5), i.e. for a sequence $(p_n) \subseteq \mathbb{N}$ and any operator $A \in \mathcal{S}_{(p_n+1)}(X)$ with eigenvalue sequence $(\lambda_n(A))$ we define

$$\det_{(p_n),(\lambda_n(A))}(\mathbb{1} - A) := \prod_{n=1}^{\infty}\left(1 - \lambda_n(A)\right)\left(\sum_{j=1}^{p_n-1}\frac{\lambda_n(A)^j}{j}\right), \tag{3.6}$$

and call $\det_{(p_n),(\lambda_n(A))}$ the **regularized determinant of type** $l^{(p_n)}$.

Remark 3.5 Let $A \in \mathcal{S}_{(p_n)}(X)$ with eigenvalue sequence $(\lambda_n(A))$. Define $\mu_1(A) := \lambda_2(A)$, $\mu_2(A) := \lambda_1(A)$ and $\mu_n(A) := \lambda_n(A)$ for all $n \geq 3$. Then $\det_{(p_n),(\mu_n(A))}(\mathbb{1}-A)$ defines a regularized determinant of type $l^{(p_n)}$. If $\lambda_1(A) \neq \lambda_2(A)$ and $p_1 \neq p_2$ then one has in general

$$\det_{(p_n),(\mu_n(A))}(\mathbb{1} - A) \neq \det_{(p_n),(\lambda_n(A))}(\mathbb{1} - A),$$

which shows that the value of the regularized determinant of type $l^{(p_n)}$ does not only depend on the operator A, but also on the order of the eigenvalue sequence of A.

Since each $\Gamma_n \leq 1$ we have, due to Theorem 2.12,

$$|\det_{(p_n),(\lambda_n(A))}(\mathbb{1} - A)| \leq \exp\Big(\sum_{n=1}^{\infty} \lambda_n(A)^{p_n}\Big) \leq \Big(\sum_{n=1}^{\infty}(c\alpha_n(A))^{p_n}\Big) < \infty,$$

with $c = 2(2e)^{\frac{1}{2}}$. As a consequence we formulate the following theorem:

Theorem 3.6 *Let $A \in \mathcal{S}_{(p_n)}$, then $\det_{(p_n),(\lambda_n(A))}(\mathbb{1} - A)$ is well defined and*

$$|\det_{(p_n),(\lambda_n(A))}(\mathbb{1} - A)| \leq \gamma_{(p_n)}(cA),$$

with $\gamma_{(p_n)}(A) := \sum_{n=1}^{\infty} \alpha_n(A)^{p_n}$ and $c = 2(2e)^{\frac{1}{2}}$.

3.1 The use of regularized determinants

In this subsection we emphasize why we use determinants defined by (3.5) or (3.6).
To clarify this let L_0 be a bounded operator and K a compact operator in the Banach space X. Let $\lambda_0 \in \Omega$, where Ω is the unbounded component of $\rho(L_0)$. We assume, that λ_0 is an eigenvalue of $L := L_0 + K$. Then there is an eigenvector f with

$$Lf = \lambda f \Leftrightarrow L_0 f + Kf = \lambda f \Leftrightarrow Kf = (\lambda\mathbb{1} - L_0)f.$$

If we define $g := (\lambda\mathbb{1} - L_0)f\big(\Leftrightarrow f = R_{L_0}(\lambda)g\big)$ we have

$$KR_{L_0}(\lambda)g = g$$

i.e. $1 \in \sigma\big(K(\lambda\mathbb{1} - L_0)^{-1}\big)$.
Now we assume that $\mathfrak{A}(X)$ is one of the ideals $\mathcal{S}_p(X), \mathcal{S}_{(p_n)}(X), \mathcal{N}(X), \Pi_p(X)$ and $\det_{\mathfrak{A}}(\mathbb{1} - \cdot)$ is the corresponding regularized determinant. If $K \in \mathfrak{A}(X)$ then, by the ideal property, also $KR_{L_0}(\lambda) \in \mathfrak{A}(X)$ for $\lambda \in \rho(L_0)$, since $R_{L_0}(\lambda)$ is a bounded operator.
Hence, for every $\lambda \in \rho(L_0)$

$$\det_{\mathfrak{A}}\big(\mathbb{1} - KR_{L_0}(\lambda)\big)$$

is well defined and by definition we have

$$d(\lambda) := \det_{\mathfrak{A}}\big(\mathbb{1} - KR_{L_0}(\lambda)\big) = 0 \Leftrightarrow 1 \in \sigma\big(KR_{L_0}(\lambda)\big) \Leftrightarrow \lambda \in \sigma_d(L). \quad (3.7)$$

The function $\det_{\mathfrak{A}}(\mathbb{1} - KR_{L_0}(\cdot))$ is called **perturbation determinant**.

With Theorem 3.4 we can formulate the following corollary.

Corollary 3.7 *Let d be defined as in (3.7), then $d(\lambda) = 0$ if and only if $\lambda \in \sigma_d(L)$ and*

1) $|d(\lambda)| \leq \exp\left(\frac{1}{2}\|KR_{L_0}(\lambda)\|_{\mathcal{N}}^2\right)$ *if $K \in \mathcal{N}(X)$ for all $\lambda \in \rho(L_0)$,*

2) $|d(\lambda)| \leq \exp\left(c_p\Gamma_p\|KR_{L_0}(\lambda)\|_{\mathcal{S}_p}^p\right)$ *for all $K \in \mathcal{S}_p(X)$ for all $p \geq 1$ and for all $\lambda \in \rho(L_0)$,*

3) $|d(\lambda)| \leq \exp\left(\Gamma_p\|KR_{L_0}(\lambda)\|_{\Pi_p}^p\right)$ *for all $K \in \Pi_p(X)$ for all $p \geq 2$ and for all $\lambda \in \rho(L_0)$*

4) $|d(\lambda)| \leq \exp\left(\frac{1}{2}\|KR_{L_0}(\lambda)\|_{\Pi_2}^2\right)$ *for all $K \in \Pi_p(X)$ for all $2 > p \geq 1$ and for all $\lambda \in \rho(L_0)$,*

where $c_p = 2(2e)^{\frac{p}{2}}$ and Γ_p as in Lemma 3.2.

As stated in the introduction we want to study the discrete spectrum of L using complex analysis methods.
d is a function the zeros of which coincide with the discrete spectrum of L. In the next section we prove that d is holomorphic if K is a $p-$summing, l^p-type, or nuclear operator. Moreover, we show that the order of a zero of d coincides with the algebraic multiplicity of the associated discrete eigenvalue. In general $\det_{(p_n),(\lambda_n(KR_{L_0}(z)))}(\mathbb{1} - KR_{L_0}(z))$ has not to be holomorphic for all $K \in \mathcal{S}_{(p_n)}(X)$ (this case is treated separately in Chapter 8).

3.2 Holomorphicity for regularized determinants of finite rank operators with fixed range

Let $\mathcal{F}(X,Y)$ denote the space of all finite rank operators from X to X with range in $Y \subseteq X$ where Y is a n-dimensional subspace of X. Since Y is finite dimensional there has to be a scalar product $\langle \cdot, \cdot \rangle$ such that Y equipped with this product is a Hilbert space[2]. For any operator $F \in \mathcal{F}(X,Y)$ we have $\sigma(F) \setminus \{0\} = \sigma(F|_Y : Y \to Y) \setminus \{0\}$.
If $A : Y \to Y$ is an operator with the matrix representation (a_{ij}) according to some basis $y_1, \ldots, y_n$ of Y, i.e $a_{ij} := \langle y_i, Ay_j \rangle$, we can rewrite the $\lceil p \rceil th$

[2]Of course, the associated Banach space norm does not have to be induced by this product, i.e. $\sqrt{\langle x, x \rangle} \neq \|x\|_X$ for some $x \in Y$

regularized determinant in terms of the matrix entries. That means

$$
\begin{aligned}
\det_{\lceil p\rceil}(\mathbb{1}-A) &= \prod_{i=1}^{n}(1-\lambda_i(A))\exp\left(\sum_{j=1}^{\lceil p\rceil-1}\frac{\lambda_i(A)^j}{j}\right)\\
&= \left(\prod_{i=1}^{n}(1-\lambda_i(A))\right)\left(\exp\left(\sum_{k=1}^{n}\sum_{j=1}^{\lceil p\rceil-1}\frac{\lambda_k(A)^j}{j}\right)\right)\\
&= \left(\prod_{i=1}^{n}(1-\lambda_i(A))\right)\left(\exp\left(\sum_{j=1}^{\lceil p\rceil-1}\sum_{k=1}^{n}\frac{\lambda_k(A^j)}{j}\right)\right)\\
&= \left(\prod_{i=1}^{n}(1-\lambda_i(A))\right)\left(\exp\left(\sum_{j=1}^{\lceil p\rceil-1}\frac{\operatorname{tr}(A^j)}{j}\right)\right)\\
&= \det\left((\delta_{ij}-a_{ij})_{i,j}\right)\exp\left(\sum_{j=1}^{\lceil p\rceil-1}\frac{\sum_{i=1}^{n}a_{ii}^{(j)}}{j}\right)\\
&= \sum_{\pi\in\mathbb{P}_n}\left(\operatorname{sgn}(\pi)\prod_{i=1}^{n}(\delta_{i,\pi(i)}-a_{i,\pi(i)})\right)\exp\left(\sum_{j=1}^{\lceil p\rceil-1}\frac{\sum_{i=1}^{n}a_{ii}^{(j)}}{j}\right),
\end{aligned}
\tag{3.8}
$$

where $a_{ii}^{(j)} := \langle y_i, A^j y_i\rangle$, is the diagonal element of the matrix representation of A^j according to the basis $y_1,\dots,y_n$ and δ_{ij} is the Kronecker symbol
This representation of the regularized determinant can be used to obtain holomorphicity for $\mathcal{F}(X,Y)$ valued, analytic functions.

Theorem 3.8 *Let $\lambda \mapsto K(\lambda) \in \mathcal{F}(X,Y)$ be analytic on a domain $\Omega \subseteq \mathbb{C}$. Then for each $p > 0$*

$$\lambda \mapsto \det_{\lceil p\rceil}(\mathbb{1}-K(\lambda))$$

is holomorphic on Ω.

Proof: Since each regularized determinant is defined by the non-zero eigenvalues it suffices to determine

$$\det_{\lceil p\rceil}((\mathbb{1}-K(\lambda))|_Y : Y \to Y)$$

where $(\mathbb{1}-K(\lambda))|_Y$ is the restriction to the finite dimensional subspace Y of X $(\sigma(K(\lambda))\setminus\{0\} = \sigma(K(\lambda)|_Y)\setminus\{0\})$. Hence, for $\det_{\lceil p\rceil}$ we can use the representation in (3.8). Moreover, if $\lambda \mapsto K(\lambda)$ is analytic then so $\lambda \mapsto K(\lambda)|_Y$, $\lambda \mapsto (\mathbb{1} - K(\lambda))\,|_Y$ and $\lambda \mapsto (K(\lambda)^m)\,|_Y$ for each $m \in \mathbb{N}$ and therefore $\lambda \mapsto \langle y_i, (\mathbb{1} - K(\lambda))|_Y y_j\rangle$, $\lambda \mapsto \langle y_i, K(\lambda)^m|_Y y_j\rangle$ are holomorphic on Ω. Hence, with (3.8) we see that $\lambda \mapsto \det_{\lceil p\rceil}(\mathbb{1} - K(\lambda))$ is a finite composition of holomorphic functions on Ω and therefore itself holomorphic on Ω. $\square$

Remark 3.9 Surely, the restriction $F(\lambda) \subseteq Y$ for all λ is not necessary. However, this restriction allows a very simple proof and is sufficient for our purposes. For completeness it should be mentioned that without this assumption, the holomorphicity of the function $\lambda \mapsto \det_1(\mathbb{1} - F(\lambda))$ has been proven in [22].

As a consequence of Theorem 3.8:

Corollary 3.10 *Let $L = L_0 + K$ be an operator on a Banach space X, L_0 a bounded operator, $K \in \mathcal{F}(X)$ and $\Omega \subseteq \rho(L_0)$. Then*

$$\lambda \mapsto d(\lambda) := \det_{\lceil p\rceil}\left(\mathbb{1} - KR_{L_0}(\lambda)\right)$$

is holomorphic on Ω.
Moreover, $\lambda_0 \in \sigma_d(L) \cap \Omega$ is of algebraic multiplicity m if and only if λ_0 is a zero of d of order m.

Proof: Note, that due to Theorem 3.8

$$\lambda \mapsto \det_{\lceil p\rceil}\left(\mathbb{1} - KR_{L_0}(\lambda)\right)$$

is holomorphic in Ω for each $p > 0$, since $\lambda \mapsto K(\lambda)$ is analytic on Ω.
Fix $p = 1$:

$$\begin{aligned} \det_1(\lambda - L) &= \det_1\left(\left(\mathbb{1} - KR_{L_0}(\lambda)\right)(\lambda - L_0)\right) \\ &= \underbrace{\det_1\left(\mathbb{1} - KR_{L_0}(\lambda)\right)}_{=d(\lambda)} \underbrace{\det_1(\lambda - L_0)}_{=:\tilde{d}(\lambda)}. \end{aligned}$$

Due to Proposition 3.1 we know that $\lambda_0 \in \Omega$ is a discrete eigenvalue of L with algebraic multiplicity m if and only if m is a zero of $\det_1(\lambda - L)$ of order m. Since $\tilde{d}$ is holomorphic and $\tilde{d}(\lambda) \neq 0$ for all $\lambda \in \Omega \subseteq \rho(L_0)$, this is equivalent to λ_0 is a zero of d of order m.
If $p > 1$ then due to (3.8)

$$d(\lambda) = \det_1\left(\mathbb{1} - KR_{L_0}(\lambda)\right)\exp\left(F(\lambda)\right),$$

where F denotes a holomorphic function on Ω.
Of course $\exp(F(\lambda)) \neq 0$ for all $\lambda \in \Omega$, and therefore, using the result for $p = 1$, $\lambda_0 \in \Omega$ is a discrete eigenvalue of L with algebraic multiplicity m if and only if λ_0 is a zero of d of order m. $\square$

3.3 Holomorphic spectral determinants

Another abstract definition of determinants defined on an operator ideal $\mathfrak{I}(X)$ is given below. To avoid confusion with the term of regularized determinants in the previous sections, this kind of determinants will be denoted by $\mathbf{det}$ instead of det.

Definition 3.11 Let $\mathbf{det} : \mathfrak{I}(X) \to \mathbb{C}$ be a map with the following properties:

1) $\mathbf{det}(\mathbb{1} - \langle \psi, \cdot \rangle f) = 1 - \langle \psi, f \rangle$ for all $\psi \in X^*, f \in X$,
2) $\mathbf{det}(\mathbb{1} - AB) = \det(\mathbb{1} - BA)$ for all $A \in \mathfrak{I}(X), B \in \mathcal{L}(X)$,
3) $\mathbf{det}\big((\mathbb{1} - A)(\mathbb{1} - B)\big) = \mathbf{det}(\mathbb{1} - A)\mathbf{det}(\mathbb{1} - B)$ for all $A, B \in \mathfrak{I}(X)$,
4) for every $A \in \mathfrak{I}(X)$ the map $\lambda \mapsto \mathbf{det}(\mathbb{1} - \lambda A)$ is an entire function.

If $\mathfrak{I}(X)$ is an ideal such that every operator in this ideal has summable eigenvalues and

$$\mathbf{det}(\mathbb{1} - A) = \det_1(\mathbb{1} - A) = \prod_{n=1}^{\infty} \big(1 - \lambda_n(A)\big) \text{ for all } A \in \mathfrak{I}(X)$$

then we call this determinant **det spectral**.

Now we give an easy condition for $\mathbf{det}$ on quasi-Banach ideals, such that $\mathbf{det}$ provides holomorphicity.

Proposition 3.12 (see Pietsch [30] p. 189, 193) *Let $(\mathfrak{A}(X), \|\cdot\|_{\mathfrak{A}})$ be a quasi-Banach ideal and* $\mathbf{det}$ *a determinant satisfying (1)-(4) in Definition 3.11. If* $\mathbf{det}$ *is continuous in 0, then* $\mathbf{det}$ *is continuous everywhere. Moreover, if*

$$\lambda \mapsto A(\lambda) \in \mathfrak{A}(X)$$

is analytic on $\Omega \subseteq \mathbb{C}$ then

$$\lambda \mapsto \mathbf{det}\big(\mathbb{1} - A(\lambda)\big)$$

is holomorphic on Ω.

Proposition 3.13 (see e.g. Pietsch [30] 4.2.24, 4.6.3, 4.6.4) *Whenever* $(\mathfrak{A}(X), \|\cdot\|_{\mathfrak{A}})$ *is a quasi-Banach ideal which is the closure of* $\mathcal{F}(X)$ *(with respect to* $\|\cdot\|_{\mathfrak{A}}$*), such that the eigenvalue sequence of each operator in* $\mathfrak{A}(X)$ *is summable, then there is a continuous spectral determinant* **det** *defined on* $\mathfrak{A}(X)$*.*

3.4 Proof of holomorphicity for regularized determinants for nuclear operators and $p-$summing operators

Since the ideal of nuclear operators is contained in the ideal $\Pi_p^{\mathcal{F}}(X)$ (Remark 2.10), it suffices to give a proof for this class of operators.
Thus one has to show that

$$\lambda \mapsto \det\nolimits_{\lceil p\rceil}\big(\mathbb{1} - K(\lambda\mathbb{1} - L_0)^{-1}\big) \tag{3.9}$$

is holomorphic on a domain $\Omega \subseteq \rho(L_0)$ for $K \in \Pi_p^{\mathcal{F}}(X)$.

Theorem 3.14 *Let* $\lambda \mapsto K(\lambda)$ *be an analytic map in* $\Omega \subseteq \mathbb{C}$ *with values in* $\Pi_p^{\mathcal{F}}(X)$*. Then* d*, defined by*

$$d(\lambda) := \det\nolimits_{\lceil p\rceil}(\mathbb{1} - K(\lambda)),$$

is a holomorphic function in Ω*.*

Proof: A proof of this theorem one can find in [23] p. 93-94. For convenience of the reader one can find an alternative proof below.
At first let us note, that

$$\Pi_p^{\lceil p\rceil}(X) := \{A_1 \ldots A_{\lceil p\rceil} \;:\; A_i \in \Pi_p^{\mathcal{F}}(X)\}$$

together with the quasi-norm

$$\|A\|_{\Pi_p^{\lceil p\rceil}} := \inf\{\|A_1\|_{\Pi_p} \cdot \ldots \cdot \|A_{\lceil p\rceil}\|_{\Pi_p}, A_i \in \Pi_p^{\mathcal{F}}(X), A = A_1 \cdot \ldots \cdot A_{\lceil p\rceil}\}$$

for $A \in \Pi_{\Pi_p}^{\lceil p\rceil}$, creates a quasi-Banach ideal of compact operators (see e.g. [30] p.27).
Since the eigenvalues of every operator in $\Pi_p^{\lceil p\rceil}(X)$ are summable ([30] 3.7.3), the space $\Pi_p^{\lceil p\rceil}(X)$ supports a continuous spectral determinant $\det_1$ (compare

Proposition 3.13). Following Proposition 3.12, $\lambda \mapsto \det_1\left(\mathbb{1} - \tilde{K}(\lambda)\right)$ is holomorphic on the domain $\Omega \subseteq \mathbb{C}$, if $\lambda \mapsto \tilde{K}(\lambda) \in \Pi_p^{\lceil p \rceil}(X)$ is analytic on the domain Ω.
Now let $\lambda \mapsto K(\lambda) \in \Pi_p^{\mathcal{F}}(X)$ be analytic on a domain Ω.

We define the entire function

$$f(z) := 1 - (1 - z)\exp\left(\sum_{k=1}^{\lceil p \rceil - 1} \frac{z^k}{k}\right).$$

This function satisfies

$$f(0) = 0$$

and the derivative is

$$f'(z) = z^{\lceil p \rceil - 1}\exp\left(\sum_{k=1}^{\lceil p \rceil - 1} \frac{z^k}{k}\right).$$

Hence, since 0 is a zero of f' of order $\lceil p \rceil - 1$ and $f(0) = 0$, there has to be an entire function g with $g(0) \neq 0$ and

$$f(z) = z^{\lceil p \rceil} g(z).$$

Then, by the Dunford functional calculus (see e.g. [14] p. 13-17) $g\big(K(\lambda)\big)$ is a bounded operator, and therefore

$$\lambda \mapsto f\big(K(\lambda)\big) = (K(\lambda))^{\lceil p \rceil}\, g\big(K(\lambda)\big)$$

is an $\Pi_p^{\lceil p \rceil}(X)$ valued analytic function. Due to the previous argumentation one can see that

$$\lambda \mapsto \det_1\big(\mathbb{1} - f\big((K(\lambda)\big)\big)$$

is holomorphic on Ω. The spectral mapping theorem for bounded operators (Theorem 1.11) shows,

$$\sigma\big(f(K(\lambda))\big) = \left\{ 1 - (1 - \mu)\left(\sum_{k=1}^{\lceil p \rceil - 1} \frac{\mu^k}{k}\right) \;:\; \mu \in \sigma\big(K(\lambda)\big) \right\}.$$

Finally,

$$\begin{aligned}
\det_1\left(\mathbb{1} - f\left(K(\lambda)\right)\right) &= \prod_{\tilde{\mu}\in\sigma(f(K(\lambda))}(1-\tilde{\mu}) \\
&= \prod_{\mu\in\sigma(K(\lambda))}\left(1-\left(1-(1-\mu)\exp\left(\sum_{k=1}^{\lceil p\rceil-1}\frac{\mu^k}{k}\right)\right)\right) \\
&= \prod_{\mu\in\sigma(K(\lambda))}(1-\mu)\left(\sum_{k=1}^{\lceil p\rceil-1}\frac{\mu^k}{k}\right) \\
&= \det_{\lceil p\rceil}\left(\mathbb{1}-K(\lambda)\right).
\end{aligned}$$

Since the left hand side of the previous equation depends holomorphically on λ (recall that $\det_1$ is continuous and spectral on $\Pi_p^{\lceil p\rceil}$), the same is true for the right hand side. $\square$

We can apply the general result in Theorem 3.14 to (3.9).

Theorem 3.15 *Let $L_0 \in \mathcal{L}(X)$, X Banach space, and $L = L_0 + K$, $K \in \Pi_p^{\mathcal{F}}(X)$. Then the determinant*

$$d(\cdot) = \det_{\lceil p\rceil}\big(\mathbb{1} - KR_{L_0}(\cdot)\big)$$

is holomorphic on $\rho(L_0)$.
Moreover there is the following connection between the algebraic multiplicity $m_\lambda(L)$ of any eigenvalue λ of L and the order $o_\lambda(d)$ of any zero of d.

$$\lambda \in \sigma_d(L) \textit{ with } m_\lambda(L) = m \Leftrightarrow \lambda \in \mathcal{Z}(d) \textit{ with } o_\lambda(d) = m,$$

where $\mathcal{Z}(d) := \{\lambda \in \rho(L_0) \,:\, d(\lambda) = 0\}$ denotes the set of zeros of d.

Proof: Since $R_{L_0}(\cdot)$ is analytic on $\rho(L_0)$, d is holomorphic on $\rho(L_0)$.
In Section 3.1 it was already mentioned, that the zeros of d coincide with the discrete spectrum of L. It remains to show that the algebraic multiplicity of any discrete eigenvalue of L is equal to its order as a zero of d.
For this let us fix an eigenvalue $\lambda_0 \in \sigma_d(L)$ and an $\epsilon > 0$, such that $B_\epsilon(\lambda_0) := \{\lambda \,:\, |\lambda - \lambda_0| \le \epsilon\} \cap \sigma_d(L) = \{\lambda_0\}$.
Next we choose a sequence $(K_n) \subseteq \mathcal{F}(X)$ with

$$\|K - K_n\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

For $L_n := L_0 + K_n$ (3.10) implies

$$\|L - L_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Theorem 1.12 and (3.11) imply that there is an $N_\epsilon \in \mathbb{N}$ with

$$\sum_{\mu \in \sigma_d(L_n) \cap B_\epsilon(\lambda_0)} m_\mu(L_n) = m_{\lambda_0}(L) \text{ for all } n \geq N_\epsilon. \tag{3.12}$$

Associated to (L_n) we have a sequence of holomorphic funtions

$$d_n(\lambda) := \det_{\lceil p \rceil}\left(\mathbb{1} - K_n R_{L_0}(\lambda)\right) \text{ for } \lambda \in \rho(L_0)$$

with $d_n(\lambda) = 0$ iff $\lambda \in \sigma_d(L_n)$. Since $K_n \in \mathcal{F}(X)$ we know that $K_n R_{L_0}(\lambda) \in \mathcal{F}(X, \mathrm{Ran}(K_n))$ and due to Corollary 3.10 there is the following connection between the eigenvalues of $L_0 + K_n$ and the zeros of d_n,

$$\mu \in \sigma_d(L_n) \text{ with } m_\mu(L_n) = m \Leftrightarrow d_n(\mu) = 0 \text{ and } o_\mu(d_n) = m. \tag{3.13}$$

Using (3.10) we can conclude that $\|KR_{L_0}(\lambda) - K_n R_{L_0}(\lambda)\| \to 0$ locally uniformly. This result implies that $d_n \to d$ locally uniformly (see Theorem 5.9). Thus we can find $N \geq N_\epsilon$ such that

$$|d_n(\lambda) - d(\lambda)| \leq |d(\lambda)|$$

for all $\lambda \in \partial B_\epsilon(\lambda_0)$ and for $n \geq N$. Rouche's Theorem (see e.g. [33], p.225) provides

$$\sum_{\mu \in \sigma_d(L_n) \cap B_\epsilon(\lambda_0)} o_\mu(d_n) = o_{\lambda_0}(d) \text{ for all } n \geq N.$$

Now using this formula, equation (3.12) and equivalence (3.13) we receive

$$o_{\lambda_0}(d) = m_{\lambda_0}(L).$$

On the other hand, if λ_0 is a zero of d, we already know that λ_0 is a discrete eigenvalue of L. Hence, by the previous arguments, the algebraic multiplicity of λ_0 as an eigenvalue of L is equal to the order of λ_0 as a zero of d. $\square$

3.5 Regularized perturbation determinants on the unbounded component of $\rho(L_0)$

In this subsection, which is based on Section 3 of the joint work [7] (M. Demuth, F. Hanauska, M. Hansmann and G. Katriel), we treat the case that a free operator L_0 is perturbed by a compact operator K. In contrast to the

former subsection we do not assume that the sequence of eigenvalues of K $(\lambda_n(K))$ satisfies any summability property, the only restriction on K is that $\lim_{n\to\infty} \alpha_n(K) = 0$, here again $\alpha_n(K)$ are the approximation numbers of the operator K.
Let $\Omega \subseteq \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be a connected, open set with $\infty \in \Omega$ and

$$\overline{\Omega} \cap \sigma(L_0) = \emptyset \quad (\text{ such that } \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\| < \infty).$$

Theorem 3.16 *Let $p > 0$ and $N \in \mathbb{N}$ such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|}$. Then there is a holomorphic function $d : \Omega \to \mathbb{C}$ with the properties:*

(i) $d(\infty) = 1$,

(ii)

$$\begin{aligned} |d(\lambda)| &\leq \exp\left(\frac{C_p\|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\|R_{L_0}(\lambda)\|)^p}\sum_{j=1}^{N}(\alpha_{N+1}(K)+\alpha_j(K))^p\right) \\ &\leq \exp\left(\frac{C_p\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|)^p}\sum_{j=1}^{N}(\alpha_{N+1}(K)+\alpha_j(K))^p\right) \end{aligned}$$

for all $\lambda \in \Omega$, with $C_p = 2(2e)^{\frac{p}{2}}\Gamma_p$ and Γ_p as in Lemma 3.2.

(iii) *$\lambda_0 \in \Omega$ is a zero of d of order m if and only if it is a discrete eigenvalue of L in Ω with algebraic multiplicity m.*

Proof: Let $p > 0$ be fixed and $N \in \mathbb{N}$ such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda\in\Omega}\|(R_{L_0}(\lambda)\|}$. For every $\eta > 0$ with $\eta < \frac{1}{\sup_{\lambda\in\Omega}\|(R_{L_0}(\lambda)\|} - \alpha_{N+1}(K)$ there is an operator $F \in \mathcal{F}(X)$ of rank at most N with

$$\|K - F\| < \alpha_{N+1}(K) + \eta.$$

Then we can estimate

$$\begin{aligned} \|(K-F)R_{L_0}(\lambda)\| &\leq \|K-F\|\|R_{L_0}(\lambda)\| \leq \big(\alpha_{N+1}(K)+\eta\big)\|R_{L_0}(\lambda)\| \\ &\leq \big(\alpha_{N+1}(K)+\eta\big)\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\| < 1. \end{aligned}$$

Therefore $1 \notin \sigma\big((K-F)R_{L_0}(\lambda)\big)$ and hence $\mathbb{1} - (K-F)R_{L_0}(\lambda)$ is invertible with

$$\|\left(\mathbb{1} - (K-F)R_{L_0}(\lambda)\right)^{-1}\| \leq \left(1 - (\alpha_{N+1}(K)+\eta)\,\|R_{L_0}(\lambda)\|\right)^{-1}. \qquad (3.14)$$

This implies that

$$\lambda \mathbb{1} - (L-F) = (\mathbb{1} - (K-F)R_{L_0}(\lambda))\,(\lambda - L_0) \tag{3.15}$$

has to be invertible, as product of invertible operators, for every $\lambda \in \Omega \setminus \{\infty\}$ and $\Omega \subseteq \hat{\rho}(L-F)$.
As a conclusion

$$d_F(\lambda) := \det_{\lceil p \rceil}(\mathbb{1} - FR_{L-F}(\lambda))$$

is well defined for every $\lambda \in \Omega$, $d_F(\infty) = 1$ and $d_F(\lambda) = 0$ of order m if and only if $\lambda \in \sigma_d(L-F+F=L)$ has the algebraic multiplicity m (see Corollary 3.10).
Moreover, Theorem 1.16 and Theorem 3.4 imply

$$|d_F(\lambda)| \leq \exp\left(2(2e)^{\frac{p}{2}}\Gamma_p \sum_{j=1}^{N} \alpha_j^p(FR_{L-F}(\lambda))\right).$$

The approximation numbers on the right hand side of the previous inequality can be estimated in the following way (see Proposition 1.15 and (3.14)):

$$\begin{aligned}
\alpha_j\left(FR_{L-F}(\lambda)\right) &= \alpha_j\left(FR_{L_0}(\lambda)\left(\mathbb{1} - (K-F)R_{L_0}(\lambda)\right)^{-1}\right) \\
&\leq \alpha_j\left(FR_{L_0}(\lambda)\right)\|\left(\mathbb{1} - (K-F)R_{L_0}(\lambda)\right)^{-1}\| \\
&\leq \frac{\alpha_j\left(FR_{L_0}(\lambda)\right)}{1 - (\alpha_{N+1}(K) + \eta)\,\|R_{L_0}(\lambda)\|}
\end{aligned}$$

And for the numerator of the right hand side of the last inequality one has:

$$\begin{aligned}
\alpha_j\left(FR_{L_0}(\lambda)\right) &= \alpha_j\left((F-K)R_{L_0}(\lambda) + KR_{L_0}(\lambda)\right) \\
&\leq \|R_{L_0}(\lambda)\|\alpha_j(F-K+K) \\
&\leq \|R_{L_0}(\lambda)\|\left(\|F-K\| + \alpha_j(K)\right) \\
&\leq \|R_{L_0}(\lambda)\|\left(\alpha_{N+1}(K) + \eta + \alpha_j(K)\right).
\end{aligned}$$

All in all we have

$$\sum_{j=1}^{N} \alpha_j\left(FR_{L-F}(\lambda)\right)^p \leq \frac{\|R_{L_0}(\lambda)\|^p \sum_{j=1}^{N}\left(\alpha_{N+1}(K) + \eta + \alpha_j(K)\right)^p}{\left(1 - (\alpha_{N+1}(K) + \eta)\,\|R_{L_0}(\lambda)\|\right)^p}.$$

The proof of this theorem ends with the following limiting argument.
There is an integer $N_0 \in \mathbb{N}$ such that $\alpha_{N_0+1}(K) < \frac{1}{\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|}$ and let l_0 denote the smallest integer such that $\frac{1}{l_0} < \frac{1}{\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|} - \alpha_{N_0+1}(K)$. For

every $l \geq l_0$ there exists a rank-N_0 operator F_l such that the holomorphic function d_{F_l}, defined as in the previous part of the proof, satisfies

$$\begin{aligned}
|d_{F_l}(\lambda)\| &\leq \exp\left(2(2e)^{\frac{p}{2}}\Gamma_p \frac{\|R_{L_0}(\lambda)\|^p \sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\frac{1}{l}+\alpha_j(K)\right)^p}{\left(1-\left(\alpha_{N+1}(K)+\frac{1}{l_0}\right)\|R_{L_0}(\lambda)\|\right)^p}\right) \\
&\leq \exp\left(2(2e)^{\frac{p}{2}}\Gamma_p \frac{\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|^p \sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\frac{1}{l}+\alpha_j(K)\right)^p}{\left(1-\left(\alpha_{N+1}(K)+\frac{1}{l_0}\right)\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|\right)^p}\right) \\
&< \infty
\end{aligned}$$

for all $\lambda \in \Omega$. Therefore, the sequence of holomorphic functions $(d_{F_l})_{l\geq l_0}$ defined on Ω is uniformly bounded.

Using Montel's theorem (see e.g. [33] Theorem 14.6), there has to be a subsequence, converging locally uniformly to a holomorphic function d. Since $d_{F_l}(\infty) = 1$ for all $l \geq l_0$, also $d(\infty) = 1$. Moreover, $\mathcal{Z}(d_{F_l}) = \sigma_d(L) \cap \Omega$ for each $l \geq l_0$ implies together with the Hurwitz' theorem (see e.g. [3]), that also $\mathcal{Z}(d) = \sigma_d(L) \cap \Omega$.

□

As an easy consequence of this theorem we can also derive a holomorphic function on the whole unbounded component of $\rho(L_0)$ if $K \in \mathcal{S}_p(X)$.

Corollary 3.17 *Let $p > 0$, $K \in \mathcal{S}_p(X)$ and $\Omega \subseteq \hat{\mathbb{C}}$ the unbounded component of $\hat{\rho}(L_0)$ with $\infty \in \Omega$. Then there exists a holomorphic function $d : \Omega \to \mathbb{C}$ with the properties:*

(i) $d(\infty) = 1$,

(ii)

$$|d(\lambda)| \leq \exp\left(C_p \|R_{L_0}(\lambda)\|^p \|K\|_{\mathcal{S}_p}^p\right) \tag{3.16}$$

for all $\lambda \in \Omega$, with $C_p = 2(2e)^{\frac{p}{2}}\Gamma_p$.

(iii) $\lambda_0 \in \Omega$ is a zero of d of order m if and only if it is a discrete eigenvalue of L in Ω with algebraic multiplicity m.

Proof: We approximate the unbounded component Ω of the resolvent set by a sequence (Ω_n) of connected open sets with

- $\infty \in \Omega_n$,
- $\overline{\Omega_n} \cap \sigma(L_0) = \emptyset$,

- $\bigcup_n \Omega_n = \Omega$.

Due to Theorem 3.16 for every Ω_n there is a holomorphic function $d_n : \Omega_n \to \mathbb{C}$ which satisfies point (i) and (iii) of Corollary 3.17 and

$$\begin{aligned}
|d_n(\lambda)| &\leq \exp\left(\frac{C_p \|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\|R_{L_0}(\lambda)\|)^p} \sum_{j=1}^{N} (\alpha_{N+1}(K)+\alpha_j(K))^p \right) \\
&\leq \exp\left(\frac{C_p \sup_{\lambda\in\Omega_n} \|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\sup_{\lambda\in\Omega_n}\|R_{L_0}(\lambda)\|)^p} \sum_{j=1}^{N} (\alpha_{N+1}(K)+\alpha_j(K))^p \right) \\
&< \infty
\end{aligned}$$

for all $\lambda \in \Omega$.
By Montel's Theorem, for Ω_1 there has to be a subsequence (n_k) such that $(d_{n_k}|_{\Omega_1})$ converges locally uniformly on Ω_1 to a holomorphic function $d_{\Omega_1} : \Omega_1 \to \mathbb{C}$. Once again using Montel's Theorem, there has to be a subsequence (n_{k_1}) of (n_k) such that $(d_{n_{k_1}}|_{\Omega_2})$ converges locally uniformly on Ω_2 to a holomorphic function $d_{\Omega_2} : \Omega_2 \to \mathbb{C}$ with the property $d_{\Omega_2}|_{\Omega_1} = d_{\Omega_1}$. Inductively, we obtain a sequence of holomorphic functions $d_{\Omega_n} : \Omega_n \to \mathbb{C}$ with $d_{\Omega_n}|_{\Omega_{n-1}} = d_{\Omega_{n-1}}$.
Hence

$$d(\lambda) := \begin{cases} d_{\Omega_1}(\lambda), & \lambda \in \Omega_1, \\ d_{\Omega_{n+1}}(\lambda), & \lambda \in \Omega_{n+1} \setminus \Omega_n \end{cases}$$

defines a holomorphic function on Ω which satisfies, due to Hurwitz' theorem (compare with the proof of Theorem 3.16), point (i) and (iii) of Corollary 3.17.
Moreover, d has the following upper bound:

$$|d(\lambda)| \leq \lim_{N\to\infty} \exp\left(\frac{C_p \|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\|R_{L_0}(\lambda)\|)^p} \sum_{j=1}^{N} (\alpha_{N+1}(K)+\alpha_j(K))^p \right).$$

It remains to show that the right hand side of (3.16) is an upper bound for the limit of the right hand side of the previous inequality.
We have to distinguish between the cases $0 < p < 1$ and $p \geq 1$. In the first case we have, using the inequality $(a+b)^p \leq a^p + b^p$ (a, b non-negative),

$$\sum_{j=1}^{N} (\alpha_{N+1}(K)+\alpha_j(K))^p \leq \sum_{j=1}^{N} \alpha_{N+1}^p(K) + \sum_{j=1}^{N} \alpha_j^p(K).$$

For the second case with $p \geq 1$ the Minkowski inequality gives:

$$\sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p \leq \left(\left(\sum_{j=1}^{N} \alpha_{N+1}^p(K) \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{N} \alpha_j^p(K) \right)^{\frac{1}{p}} \right)^p .$$

Since $j \mapsto \alpha_j(K)^p$ is non-increasing, independent of the choice of $p > 0$, and $(\alpha_j(K)) \in l^p(\mathbb{N})$ one has

$$\sum_{j=1}^{N} \alpha_{N+1}^p(K) = N\alpha_{N+1}^p(K) \overset{N\to\infty}{\longrightarrow} 0.$$

Indeed, there are the estimates

$$\begin{aligned} 2N\alpha_{2N}^p(K) &= 2\sum_{m=N+1}^{2N} \alpha_{2N}^p(K) \leq 2\sum_{m=N+1}^{2N} \alpha_m^p(K) \overset{N\to\infty}{\longrightarrow} 0, \\ (2N+1)\alpha_{2N+1}^p(K) &= \alpha_{2N+1}^p(K) + 2\sum_{m=N+1}^{2N} \alpha_{2N+1}^p(K) \\ &\leq \alpha_{2N+1}^p(K) + 2\sum_{m=N+1}^{2N} \alpha_m^p(K) \overset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence

$$\lim_{N\to\infty} \sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p \leq \sum_{j=1}^{\infty} \alpha_j^p(K) = \|K\|_{\mathcal{S}_p}^p.$$

Recall, that $\lim_{N\to\infty} \alpha_N(K) = 0$ and therefore

$$\lim_{N\to\infty} \frac{C_p \|R_{L_0}(\lambda)\|^p}{(1 - \alpha_{N+1}(K)\|R_{L_0}(\lambda)\|)^p} = C_p \|R_{L_0}(\lambda)\|^p.$$

$\square$

Chapter 4

Collection of used results from complex analysis

In the previous chapter we proved the existence of holomorphic functions, the zeros of which coincide with the discrete spectrum of the operator $L := L_0 + K$ on unbounded components of $\rho(L_0)$, where L_0 is bounded and K belongs to one of the operator ideals introduced in Chapter 2.
In general, in any case there is a holomorphic function with this property, since the Weierstraß factorization theorem gives us a construction of such a function (see e.g. Rudin [33] p. 293-296). The advantage of our construction with regularized determinants is, that we prove not only the existence of such a function but derive also an upper bound in terms of known quantities. Thus we can apply general results on the number of zeros of holomorphic functions satisfying certain bounds, to obtain informations about the distribution of its zeros and so on the distribution of the eigenvalues of L. One of these results is Jensen's identity for holomorphic functions on the open unit disc. Since our functions of interest are defined on subsets Ω of the unbounded component of the resolvent set of L_0 one has to transform Ω, via a conformal map, to the open unit disc. Due to the Riemann mapping theorem (see e.g. [33] p. 274), such a conformal map exists if $\Omega \subsetneq \mathbb{C}$ is simply connected, or if $\Omega \subseteq \hat{\mathbb{C}}$ is simply connected in the extended plane with $\infty \in \Omega$ and Ω has more than one boundary point (see e.g. [31] p. 5).

4.1 Jensen's identity

Theorem 4.1 *If f is holomorphic on the open unit disc and $f(0) \neq 0$ then for $0 < r < 1$*

$$\log|f(0)| + \sum_{\substack{w \in \mathcal{Z}(f) \\ |z| \leq r}} \log\left|\frac{r}{w}\right| = \log|f(0)| + \int_0^r \frac{n(f,s)}{s} ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})| dt.$$

In particular, if f is normed by $|f(0)| = 1$ one has

$$\sum_{w \in \mathcal{Z}(f), |z| \leq r} \log\left|\frac{r}{w}\right| = \int_0^r \frac{n(f,s)}{s} ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})| dt,$$

where $n(f,s) := \#\{z \in \mathbb{D} : f(z) = 0, |z| \leq s\}$ denotes the number of zeros of f with modulus less than s and $\mathcal{Z}(f)$ denotes the set of zeros of f.

Proof:[see e.g. [33] p.299-300]. $\square$

An easy application of Jensen's identiy is the rate of accumulation of the zeros of holomorphic functions f with the following boundary condition.

Corollary 4.2 ([18] p. 31) *Let f be holomorphic on the open unit disc with $|f(0)| = 1$. If*

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \log|f(re^{it})| dt < \infty \tag{4.1}$$

then f satisfies the so called Blaschke condition

$$\sum_{z \in \mathcal{Z}(f)} (1 - |z|) < \infty.$$

Here again $\mathcal{Z}(f)$ denotes the set of zeros of the function f.

Proof: One can apply Jensen's identity and thus

$$\sum_{z \in \mathcal{Z}(f), |z| \leq r} \log\frac{r}{|z|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})| dt.$$

The left hand side of this equality is monotone increasing, with respect to $r \to 1$ and therefore bounded by the supremum of the right handside. Using the inequality $\log|x| \leq |x| - 1$ we have

$$\sum_{z \in \mathcal{Z}(f)} (1 - |z|) \leq \sum_{z \in \mathcal{Z}(f)} \log\frac{1}{|z|} \leq \frac{1}{2\pi} \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \log|f(re^{it})| dt < \infty.$$

$\square$

Remark 4.3 Note, if f is holomorphic on the open unit disc and uniformly bounded, then (4.1) is true with

$$\sup_{0\leq r<1}\int_{-\pi}^{\pi}\log|f(re^{it})|dt \leq 2\pi\log\|f\|_\infty,$$

and therefore

$$\sum_{z\in\mathcal{Z}(f),|z|\leq r}(1-|z|) \leq \log\|f\|_\infty.$$

4.2 A theorem of Hansmann and Katriel

In general, holomorphic functions on the open unit disc do not satisfy conditions like (4.1). A more general result, depending on the bounds on f is due to Hansmann and Katriel [20], which is an extension of a theorem by Borichev, Golinskii and Kupin [2] (see also [12]).

Theorem 4.4 *Let f be holomorphic on $\mathbb{D}$ with $|f(0)| = 1$ and*

$$|f(z)| \leq \exp\left(\frac{K|z|^\gamma}{(1-|z|)^\alpha\prod_{j=1}^N|z-z_j|^{\beta_j}}\right),$$

where $z_j \in \partial\mathbb{D}$, $\alpha, \gamma, \beta_j \geq 0$ with a fixed constant K. Then for every $\epsilon, \tau > 0$ the following holds: If $\alpha > 0$ then

$$\sum_{z\in\mathcal{Z}(f)}\frac{(1-|z|)^{\alpha+1+\tau}}{|z|^{(\gamma-\epsilon)_+}}\prod_{j=1}^N|z-z_j|^{(\beta_j-1+\tau)_+} \leq C(\alpha,\beta_i,\gamma,z_i,\epsilon,\tau)K.$$

If $\alpha = 0$ then

$$\sum_{z\in\mathcal{Z}(f)}\frac{(1-|z|)}{|z|^{(\gamma-\epsilon)_+}}\prod_{j=1}^N|z-z_j|^{(\beta_j-1+\tau)_+} \leq C(\beta_i,\gamma,z_i,\epsilon,\tau)K.$$

where $C(\dots)$ denotes a constant only depending on $b_i, z_i, \gamma, \epsilon, \tau, \alpha$ and $(x)_+ := \max(0, x)$ for $x \in \mathbb{R}$.

4.3 Conformal maps and distortion theorems

As already mentioned in the beginning of this chapter the holomorphic functions, introduced in Chapter 3, are defined on subsets Ω of the extended

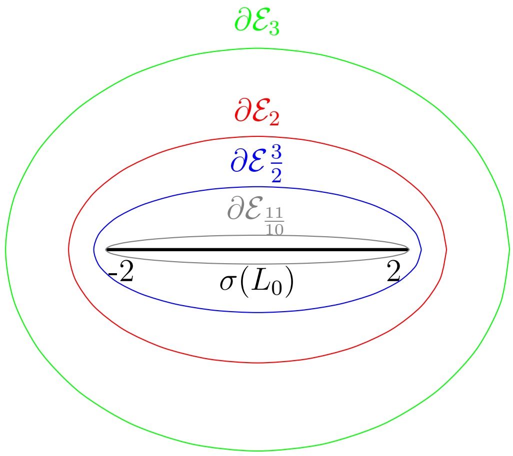

Figure 4.1: $\partial\mathcal{E}_r$ for $r = 3, 2, \frac{3}{2}, \frac{11}{10}$. $\mathcal{E}_r$ is the unbounded component of $\mathbb{C} \setminus \partial\mathcal{E}_r$.

resolvent set $\hat{\rho}(L_0)$.
Nevertheless, one can apply the results of Theorem 4.1 and Theorem 4.4 if Ω is conformal to $\mathbb{D}$.
Therefore, below there are some examples of simply connected open sets with explicitly known conformal maps $\phi : \Omega \to \mathbb{D}$ normed to $\phi(\infty) = 0$.
A very simpe example is $\Omega_1 := (s\mathbb{D})^c \cup \{\infty\}$ for some $s > 0$. Then a conformal map $\phi_1 : \Omega_1 \to \mathbb{D}$, with $\phi_1(\infty) = 0$ is given by

$$\phi_1(z) := \frac{s}{z}, \quad z \in \Omega_1 \setminus \{\infty\} = (s\mathbb{D})^c,$$
$$\phi_1^{-1}(w) = \frac{s}{w}, \quad w \in \mathbb{D} \setminus \{0\}.$$

Another example is $\Omega_2 := \mathcal{E}_s := \{\frac{1}{r}e^{i\theta} + re^{-i\theta}, r > s, \theta \in [-\pi, \pi]\} \cup \{\infty\}$ with $s \geq 1$, the unbounded component of an ellipse. A conformal map $\phi_2 : \Omega_2 \to \mathbb{C}$ with $\phi_2(\infty) = 0$ is given by

$$\phi_2(z) = s\frac{z + \sqrt{z^2 - 4}}{2}, \quad z \in \Omega_2 \setminus \{\infty\},$$
$$\phi_2^{-1}(w) = \frac{1}{s}w + s\frac{1}{w}, \quad w \in \mathbb{D} \setminus \{0\}.$$

As one can see in Figure 4.1

$$\bigcup_{s>1} \mathcal{E}_s = \hat{\mathbb{C}} \setminus [-2, 2]$$

and therefore, for every $\epsilon > 0$ there is an $s_\epsilon > 1$ such that

$$\mathrm{dist}(\mathcal{E}_{s_\epsilon}, [-2,2]) < \epsilon.$$

Another important case is $\Omega_3 := \hat{\mathbb{C}} \setminus [a,b]$. Note that, if $[a,b] = [-2,2]$, Ω_3 coincides with Ω_2 if $s = 1$. A conformal map $\phi_3 : \Omega_3 \to \mathbb{D}$ with $\phi_3(\infty) := 0$ is given by

$$\phi_3(z) = \frac{\frac{4(z-a)-2(b-a)}{b-a} + \sqrt{\left(\frac{4(z-a)-2(b-a)}{b-a}\right)^2 - 4}}{2}, \quad z \in \Omega_3 \setminus \{\infty\},$$
$$\phi_3^{-1}(w) = \frac{b-a}{4}\left(w + \frac{1}{w} + 2\right) + a, \quad w \in \mathbb{D} \setminus \{0\}.$$

The next theorem (which is due to Hansmann [18] Lemma 3.2.1) gives an important geometric interpretation of ϕ_3.

Theorem 4.5 ([18] Lemma 3.2.1) *Let $z \in \Omega_3$ with $z = \phi_3^{-1}(w)$, i.e. $z = \frac{b-a}{4}\left(w + \frac{1}{w} + 2\right) + a$ with $w \in \mathbb{D}$ then*

$$\frac{b-a}{8}\frac{|w^2-1|(1-|w|)}{|w|} \le \mathrm{dist}(z,[a,b]) \le \frac{(b-a)(1+\sqrt{2})}{8}\frac{|w^2-1|(1-|w|)}{|w|}. \tag{4.2}$$

One other kind of a simply connected set, defined as the unbounded component of an ellipse, is given by

$$\Omega_4 := \tilde{\mathcal{E}}_s := \left\{re^{i\theta} + \frac{s}{re^{i\theta}}, 0 < r < 1, \theta \in [-\pi,\pi]\right\} \cup \{\infty\}.$$

A conformal map $\phi_4 : \Omega_4 \to \mathbb{D}$ with $\phi_4(\infty) := 0$ is given by

$$\phi_4(z) = \frac{z + \sqrt{z^2 - 4s}}{2}, \; z \in \Omega_4 \setminus \{\infty\},$$
$$\phi_4^{-1}(w) = w + \frac{s}{w}, \; w \in \mathbb{D} \setminus \{0\}.$$

Also in this case we have

$$\bigcup_{s>1} \tilde{\mathcal{E}}_s = \mathbb{C} \setminus [-2,2]$$

as Figure 4.2 suggests, i.e.

$$\sup_{\lambda \in \partial\tilde{\mathcal{E}}_s} \mathrm{dist}(\lambda, [-2,2]) \xrightarrow{s\to 1} 0.$$

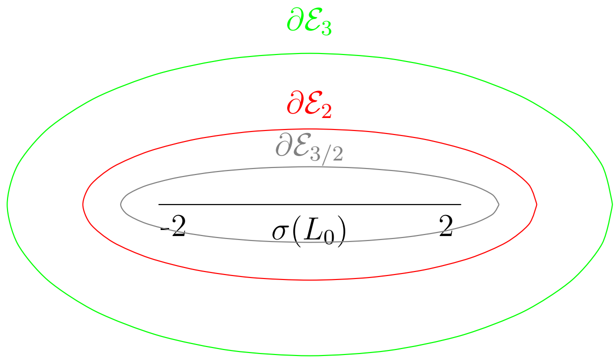

Figure 4.2: $\partial\tilde{\mathcal{E}}_s$ for $s = 3, 2, \frac{3}{2},$. $\tilde{\mathcal{E}}_s$ is the unbounded component of $\mathbb{C} \setminus \partial\tilde{\mathcal{E}}_s$.

This statemant follows by the theorem below, with a more explicit asymptotic bound of the quantity

$$\operatorname{dist}(\lambda, [-2, 2]),\ \lambda \in \partial\tilde{\mathcal{E}}_s.$$

Theorem 4.6 *Let $s > 1$ and $\lambda \in \partial\tilde{\mathcal{E}}_s$, i.e. there is a $t \in [0, 2\pi)$ with $\lambda = e^{it} + se^{-it}$, then*

$$\frac{(s-1)^{\frac{3}{2}}}{s^{\frac{1}{2}}} \leq \operatorname{dist}(\lambda, [-2, 2]) \leq s - 1.$$

Proof: It is useful to rewrite λ in terms of cosine and sine, i.e.

$$\lambda = (s + 1)\cos(t) + i(1 - s)\sin(t).$$

Consider the case $\operatorname{Re}(\lambda) \in [-2, 2]$, i.e. $\frac{-2}{s+1} \leq \cos(t) \leq \frac{2}{s+1}$, then

$$\operatorname{dist}(\lambda, [-2, 2]) = \operatorname{Im}(\lambda) = (s - 1)\sin(t) = (s - 1)(1 - \cos^2(t))^{\frac{1}{2}}.$$

As a consequence

$$\frac{(s-1)^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}{s+1} = (s-1)\left(1 - \frac{4}{(s+1)^2}\right)^{\frac{1}{2}} \leq \operatorname{dist}(\lambda, [-2, 2]) \leq s - 1.$$

If $\operatorname{Re}(\lambda) \notin [-2, 2]$ it is sufficient by symmetry to concentrate on $2 < \operatorname{Re}(\lambda) < s + 1$ (i.e. $\frac{2}{s+1} < \cos(t) < 1$), and therefore

$$\begin{aligned}\operatorname{dist}(\lambda, [-2, 2])^2 =& |\lambda - 2|^2 = (\operatorname{Re}(\lambda) - 2)^2 + \operatorname{Im}(\lambda)^2 \\ &= ((s + 1)\cos(t) - 2)^2 + (s - 1)^2(1 - \cos^2(t)).\end{aligned}$$

For $x = \cos(t)$ we define

$$f(x) = ((s+1)x-2)^2 + (s-1)^2(1-x^2), \ \frac{2}{s+1} < x < 1.$$

The only critical point of f is $x = \frac{s+1}{2s} \in \left(\frac{2}{s+1}, 1\right)$ (i.e. $f'(x) = 0$).
Since

$$f\left(\frac{s+1}{2s}\right) = \frac{(s-1)^3}{s} \leq f\left(\frac{2}{s+1}\right) = \frac{(s-1)^3(s+3)}{(s+1)^2} \leq f(1) = (s-1)^2$$

f has its minimum in $\frac{s+1}{2s}$ and its maximum in 1, and one can enclose $\mathrm{dist}(\lambda, [-2,2])$ in the following way

$$\frac{(s-1)^{\frac{3}{2}}}{s^{\frac{1}{2}}} \leq \mathrm{dist}(\lambda, [-2,2]) \leq s-1, \text{ if } \lambda \in \partial\tilde{\mathcal{E}}_s, \mathrm{Re}(\lambda) > 2.$$

Moreover, since $\frac{(s-1)^{\frac{3}{2}}}{s^{\frac{1}{2}}} < \frac{(s-1)^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}{(s+1)}$ for all $s > 1$, it follows

$$\frac{(s-1)^{\frac{3}{2}}}{s^{\frac{1}{2}}} \leq \mathrm{dist}(\lambda, [-2,2]) \leq s-1, \text{ if } \lambda \in \partial\tilde{\mathcal{E}}_s.$$

$\square$

A more general result, concerning the geometric interpretation of conformal maps is the Koebe distortion theorem (see e.g. [31] p.9.)

Theorem 4.7 *Let $\phi : \mathbb{D} \to \Omega$ be conformal. Then*

$$\frac{1}{4}\left|\phi'(w)(1-|w|)\right| \leq \mathrm{dist}(\phi(w), \partial\Omega) \leq 2\left|\phi'(w)(1-|w|)\right|$$

for all $w \in \mathbb{D}$.

Chapter 5

General results for bounded operators

In this chapter the results of the previous chapters, in particular Chapter 3 and Chapter 4, will be applied to derive statements on the distribution of the discrete spectrum of

$$L = L_0 + K,$$

where L_0 is a bounded and K a compact operator.
Here we consider very general operators. In Chapter 7 we treat more concrete examples.

5.1 The number of discrete eigenvalues and Lieb-Thirring type inequalities

The first part of this section, in particular Theorem 5.1 and the eigenvalue estimates in complements of discs, is a continuation of Section 3.5 and is based on the joint work [7] (M. Demuth, F. Hanauska, M. Hansmann, G. Katriel).

5.1.1 Eigenvalues in simply connected regions

Let K be a compact operator which is the uniform limit of finite rank operators and let $\Omega \subseteq \hat{\rho}(L_0)$ be an open set with $\infty \in \Omega$ and $\overline{\Omega} \cap \sigma(L_0) = \emptyset$.
For the sake of simplicity let us repeat the result in Theorem 3.16.
Let $p > 0$ and $N \in \mathbb{N}$ such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda \in \Omega} \|R_{L_0}(\lambda)\|}$. Then there is a holomorphic function $d : \Omega \to \mathbb{C}$ with the properties:

(i) $d(\infty) = 1$,

(ii)

$$\begin{aligned}|d(\lambda)| &\leq \exp\left(\frac{C_p \|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\|R_{L_0}(\lambda)\|)^p}\sum_{j=1}^N (\alpha_{N+1}(K)+\alpha_j(K))^p\right) \\ &\leq \exp\left(\frac{C_p \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|)^p}\sum_{j=1}^N (\alpha_{N+1}(K)+\alpha_j(K))^p\right)\end{aligned}$$

for all $\lambda \in \Omega$, with $C_p = 2(2e)^{\frac{p}{2}}\Gamma_p$, Γ_p taken from Lemma 3.2.

(iii) $\lambda_0 \in \Omega$ is a zero of d of order m if and only if it is a discrete eigenvalue of L in Ω with algebraic multiplicity m.

In the following we additionally assume that Ω is a simply connected set (in the extended plane) with more than one boundary point, i.e. conformal to the open unit disc. Let $\phi : \Omega \to \mathbb{D}$ be such a conformal map with $\phi(\infty) = 0$. Then

$$h := d \circ \phi^{-1}$$

is holomorphic on $\mathbb{D}$ and $h(0) = d(\phi^{-1}(0)) = d(\infty) = 1$. Therefore, due to Jensen's identiy (Theorem 4.1) we have

$$\int_0^1 \frac{n(h,s)}{s} ds \leq \log \|h\|_\infty. \tag{5.1}$$

Now, take $\Omega_1 \subseteq \Omega$ and denote (see Figure 5.1)

$$r_\Omega(\Omega_1) := \sup_{z\in\Omega_1} |\phi(z)|. \tag{5.2}$$

The next theorem provides estimates on the number $\mathcal{N}_L(\Omega_1)$ of discrete eigenvalues of L in Ω_1 involving the quantity $r_\Omega(\Omega_1)$.

Theorem 5.1 *Let $p > 0$ and let $\Omega \subseteq \hat{\rho}(L_0)$ be open, simply connected with more than one boundary point, with $\infty \in \Omega$ and $\overline{\Omega} \cap \sigma(L_0) = \emptyset$. If $\Omega_1 \subseteq \Omega$ is such that $0 < r_\Omega(\Omega_1) < 1$ then the following hold:*

(i) If N is such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|}$, then

$$\mathcal{N}_L(\Omega_1) \leq \frac{C_p \sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|^p}{\log\frac{1}{r_\Omega(\Omega_1)}(1-\alpha_{N+1}(K)\sup_{\lambda\in\Omega}\|R_{L_0}(\lambda)\|)^p}\sum_{j=1}^N (\alpha_{N+1}(K)+\alpha_j(K))^p,$$

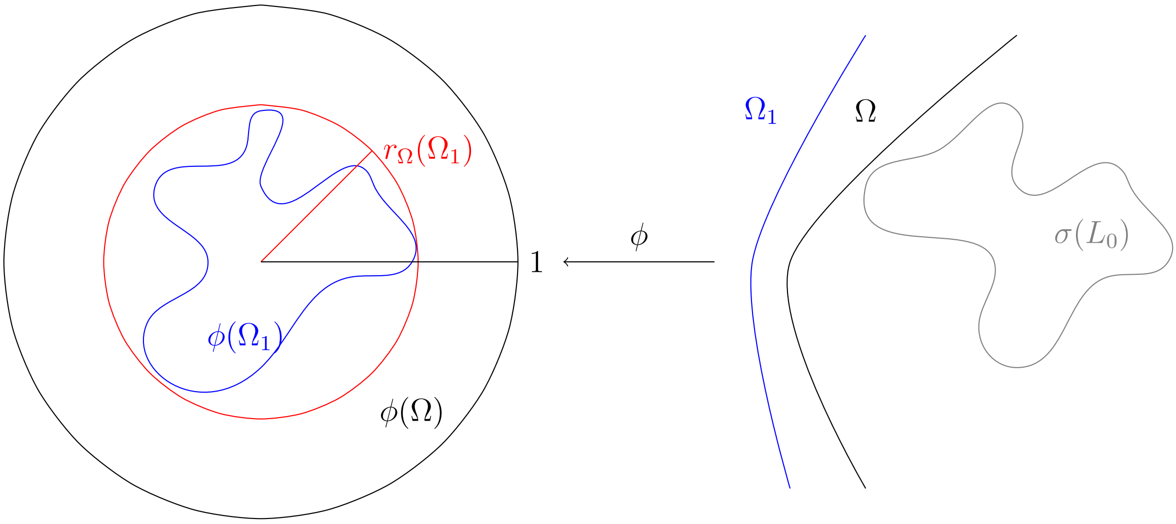

Figure 5.1: Illustration of the quantity $r_\Omega(\Omega_1)$.

(ii) if $K \in \mathcal{S}_p(X)$ *then*

$$\mathcal{N}_L(\Omega_1) \leq \frac{C_p \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|^p}{\log \frac{1}{r_\Omega(\Omega_1)}} \|K\|^p_{\mathcal{S}_p}.$$

Proof: We can estimate the left hand side of (5.1) from below.

$$\log \|h\|_\infty \geq \int_0^1 \frac{n(h,s)}{s} ds \geq \int_{r_{\Omega_1}(\Omega)}^1 \frac{n(h, r_{\Omega_1}(\Omega))}{s} ds = n(h, r_{\Omega_1}(\Omega)) \log \frac{1}{r_{\Omega_1}(\Omega)}. \tag{5.3}$$

Since $|\phi(\omega)| \leq r_{\Omega_1}(\Omega)$ for all $\omega \in \Omega_1$ we have (see Figure 5.1)

$$\mathcal{N}_L(\Omega_1) = \#\left(\sigma_d(L) \cap \Omega_1\right) = \#\{w \in \phi(\Omega_1) \,:\, h(w) = 0\} \leq n(h, r_{\Omega_1}(\Omega)), \tag{5.4}$$

and by Theorem 3.16

$$\log \|h\|_\infty \leq \frac{C_p \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|^p}{(1 - \alpha_{N+1}(K) \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|)^p} \sum_{j=1}^N (\alpha_{N+1}(K) + \alpha_j(K))^p. \tag{5.5}$$

(5.3), (5.4) and (5.5) together show the validity of (i).
To prove (ii) one has to use the inequality

$$\lim_{N\to\infty} \frac{C_p \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|^p}{(1-\alpha_{N+1}(K)\sup_{\lambda\in\Omega} \|(R_{L_0}(\lambda)\|)^p} \sum_{j=1}^{N} (\alpha_{N+1}(K)+\alpha_j(K))^p \leq \frac{C_p \sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|^p}{\log \frac{1}{r_\Omega(\Omega_1)}} \|K\|_{\mathcal{S}_p}^p$$

which was already used in the proof of Corollary 3.17. $\square$

The result in Theorem 5.1 is very general. Applying it to bound the number of eigenvalues in specific sets requires computing $r_\Omega(\Omega_1)$, which is generally hard, since of the missing knowledge on the conformal map ϕ. In Section 5.1.2 we will deal with the number of eigenvalues outside a disk and in Section 5.1.3 in the unbounded component of the complement of an ellipse.

5.1.2 Eigenvalues in the complement of discs

We take $s > t > \|L_0\|$ and

$$\Omega := (t\mathbb{D})^c \cup \{\infty\}, \quad \Omega_1 := (s\mathbb{D})^c.$$

Then Ω is simply connected in the extended plane and $\infty \in \Omega$. Moreover, we can derive a useful upper bound on the resolvent of L_0 on Ω.

$$\|R_{L_0}(\lambda)\| \leq |\lambda|^{-1} \left\| \left(\mathbb{1} - \frac{L_0}{\lambda}\right)^{-1} \right\| \leq |\lambda|^{-1} \left(1 - \frac{\|L_0\|}{|\lambda|}\right)^{-1} < \frac{1}{t - \|L_0\|}$$

for $\lambda \in \Omega$. The correct conformal map $\phi : \Omega \to \mathbb{D}$ is given by $\phi(z) = \frac{t}{z}$ for $z \in (t\mathbb{D})^c$ and $\phi(\infty) = \infty$, and so $r_\Omega(\Omega_1) = \frac{t}{s}$.
Therefore, if

$$\|L_0\| + \alpha_{N+1}(K) < t < s \text{ which implies } \alpha_{N+1}(K) < t - \|L_0\| \leq \frac{1}{\sup_{\lambda\in\Omega} \|R_{L_0}(\lambda)\|}$$

we obtain from Theorem 5.1 the inequality

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq \frac{C_p}{\log \frac{s}{t} \left(t - (\|L_0\| + \alpha_{N+1}(K))\right)^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p. \tag{5.6}$$

The right hand side depends on the value t. That means we have to minimize the right hand side of this inequality, by maximizing the smooth function

$$f(t) := \left(\log \frac{s}{t}\right)(t-A)^p$$

with $A := \|L_0\| + \alpha_{N+1}(K)$ on the intervall (A, s). Since, f is strictly positive and vanishes in the points A and s, the maximum has to be in the interior of this interval and it is $t^* \in (A, s)$ with $f'(t^*) = 0$, that is

$$\begin{aligned}
&-\frac{1}{t^*}(t^*-A)^p + p\left(\log \frac{s}{t^*}\right)(t^*-A)^{p-1} = 0 \\
\Leftrightarrow & 1 - \frac{A}{t^*} = p\left(\log \frac{s}{t^*}\right) \Leftrightarrow \frac{A}{pt^*}e^{\frac{A}{pt^*}} = \frac{A}{ps}e^{\frac{1}{p}} \\
\Leftrightarrow & \frac{A}{pt^*} = W\left(\frac{A}{ps}e^{\frac{1}{p}}\right) \Leftrightarrow t^* = \frac{A}{pW\left(\frac{A}{ps}e^{\frac{1}{p}}\right)},
\end{aligned}$$

where W denotes the Lambert W-function $W : [0, \infty) \to [0, \infty)$, implicitly given by the equality

$$W(x)e^{W(x)} = x. \tag{5.7}$$

Thus

$$\begin{aligned}
f(t^*) &= \log\left(\frac{ps}{A}W\left(\frac{A}{ps}e^{\frac{1}{p}}\right)\right)\left(\frac{A}{pW\left(\frac{A}{p}se^{\frac{1}{p}}\right)} - A\right)^p \\
&= \left(\frac{1}{pW\left(\frac{1}{p}e^{\frac{1}{p}}\frac{A}{s}\right)} - 1\right)^{p+1} W\left(\frac{1}{p}e^{\frac{1}{p}}\frac{A}{s}\right)\frac{A^p}{s^p}s^p.
\end{aligned}$$

Hence we can deduce from that:

Theorem 5.2 *Let $p > 0$, $s > \|L_0\|$ then:*

(i) If $N \in \mathbb{N}$ such that $\alpha_{N+1}(K) < s - \|L_0\|$,

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \le \frac{C_p}{s^p}\Phi_p\left(\frac{\|L_0\| + \alpha_{N+1}(K)}{s}\right)\sum_{j=1}^{N}\left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p,$$

(ii) if $K \in \mathcal{S}_p(X)$ then

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \le \frac{C_p}{s^p}\Phi_p\left(\frac{\|L_0\|}{s}\right)\|K\|_{\mathcal{S}_p}^p$$

with

$$\Phi_p(x) := \frac{\left(W(\frac{1}{p}e^{\frac{1}{p}}x)\right)^p}{\left(\frac{1}{p} - W\left(\frac{1}{p}e^{\frac{1}{p}}x\right)\right)^{p+1} x^p}$$

for $x \in (0,1)$.

A more precise assertion, not involving the implicit function Φ_p, is given by the following corollary.

Corollary 5.3 *Let* $p > 0$, $s > \|L_0\|$ *then:*

(i) If $N \in \mathbb{N}$ *such that* $\alpha_{N+1}(K) < s - \|L_0\|$, *then*

$$\mathcal{N}_L\left(s\mathbb{D})^c\right) \leq \frac{C_p(p+1)^{p+1}}{p^p} \frac{s}{(s - (\|L_0\| + \alpha_{N+1}(K)))^{p+1}} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p,$$

(ii) if $K \in \mathcal{S}_p(X)$ *then*

$$\mathcal{N}_L\left(s\mathbb{D})^c\right) \leq \frac{C_p(p+1)^{p+1}}{p^p} \frac{s}{(s - \|L_0\|)^{p+1}} \|K\|_{\mathcal{S}_p}^p$$

Proof: Recall from (5.6), if $\|L_0\| + \alpha_{N+1}(K) < t < s$, the number of eigenvalues in $(s\mathbb{D})^c$ can be estimated by

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq \frac{C_p}{\log\left(\frac{s}{t}\right)\left(t - \|L_0\| + \alpha_{N+1}(K)\right))^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p. \tag{5.8}$$

Note the two facts:

(a) If $\|L_0\| + \alpha_{N+1}(K) < t < s$ then

$$\log\left(\frac{s}{t}\right) = -\log\left(\frac{t}{s}\right) \geq -\left(\frac{t}{s} - 1\right) = 1 - \frac{t}{s}. \tag{5.9}$$

(b) For every $b > 1$

$$t(s) := \frac{b-1}{b}s + \frac{1}{b}\left(\|L_0\| + \alpha_{N+1}(K)\right)$$

satisfies

$$\|L_0\| + \alpha_{N+1}(K) < t(s) < s.$$

Combining (5.8) and (5.9) we obtain

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq \frac{C_p}{\left(1-\frac{t}{s}\right)\left(t-(\|L_0\|+\alpha_{N+1}(K))\right)^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p, \tag{5.10}$$

for every $\|L_0\|+\alpha_{N+1}(K) < t < s$. In particular for $t = t(s)$ inequality (5.10) is

$$\begin{aligned}
\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq & \frac{1}{\left(1-\frac{\frac{b-1}{b}s+\frac{1}{b}(\|L_0\|+\alpha_{N+1}(K))}{s}\right)} \qquad (5.11)\\
& \times \frac{C_p}{\left(\frac{b-1}{b}s+\frac{1}{b}\left(\|L_0\|+\alpha_{N+1}(K)\right)-\left(\|L_0\|+\alpha_{N+1}(K)\right)\right)^p} \\
& \times \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p \\
\leq & \frac{C_p s}{\frac{1}{b}(s-(\|L_0\|+\alpha_{N+1}(K)))\left(\frac{b-1}{b}\right)^p\left(s-(\|L_0\|+\alpha_{N+1}(K))\right)^p} \\
& \times \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p \\
\leq & \frac{C_p b^{p+1}}{(b-1)^p} \frac{s}{\left(s-(\|L_0\|+\alpha_{N+1}(K))\right)^{p+1}} \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p. \qquad (5.12)
\end{aligned}$$

The function $b \mapsto \frac{b^{p+1}}{(b-1)^p}$ has its minimal value at $b = p+1$, and therefore (5.12) becomes

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq \frac{C_p(p+1)^{p+1}}{p^p} \frac{s}{\left(s-(\|L_0\|+\alpha_{N+1}(K))\right)^{p+1}} \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p.$$

$\square$

Remark 5.4 Since $\lim_{x\to 0}\Phi_p(x) = pe$ we can derive also an explicit upper bound for the number of eigenvalues if $L_0 = 0$, i.e. $L = K$:

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq \frac{epC_p}{s^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p.$$

Moreover, if $K \in \mathcal{S}_p(X)$ then

$$\mathcal{N}_L\left((s\mathbb{D})^c\right) \leq \frac{epC_p}{s^p}\|K\|^p_{\mathcal{S}_p}.$$

5.1.3 Eigenvalues in the unbounded component of complements of ellipses

Now we want to treat the number of eigenvalues in the unbounded component of an ellipse.
We further assume that the spectrum of L_0 is a real intervall, i.e. $\sigma(L_0) = [a, b] \subseteq \mathbb{R}$, and that there is an $M \geq 1$ such that

$$\|R_{L_0}(\lambda)\| \leq \frac{M}{\operatorname{dist}(\lambda, [a, b])}.$$

Without loss of generality we assume that $[a, b] = [-2, 2]$. This is possible by Theorem 1.8 (the spectral mapping theorem). In fact, if we use the linear transform $y(z) := \frac{4}{b-a}z - 2\frac{b+a}{b-a}$ we have $y([a, b]) = [-2, 2]$ and $\sigma(y(L_0)) = y(\sigma(L_0))$.
In Section 4 were introduced two different kinds of ellipses.

$$\mathcal{E}_t := \left\{\frac{1}{r}e^{i\theta} + re^{-i\theta} : r > t, \theta \in [-\pi, \pi]\right\} \cup \{\infty\} \subseteq \rho(\hat{L}_0)$$

and

$$\tilde{\mathcal{E}}_t := \left\{re^{i\theta} + \frac{t}{r}e^{-i\theta}, 0 < r < 1, \theta \in [-\pi, \pi]\right\} \cup \{\infty\}.$$

$\mathcal{E}_t$ is conformal to $\mathbb{D}$, via the conformal map $\phi_{2,t} : \mathcal{E}_t \to \mathbb{D}$ defined by

$$\phi_{2,t}(z) = t\frac{z + \sqrt{z^2 - 4}}{2}, \quad \phi_{2,t}(\infty) = 0$$

and the inverse is given by

$$\phi_{2,t}^{-1}(w) := \frac{1}{t}w + t\frac{1}{w}, \quad \phi_{2,t}^{-1}(0) = \infty.$$

$\tilde{\mathcal{E}}_t$ is conformal to $\mathbb{D}$, via the conformal map $\phi_{4,t} : \tilde{\mathcal{E}}_t \to \mathbb{D}$ defined by

$$\phi_{4,t}(z) := \frac{z + \sqrt{z^2 - 4t}}{2}, \quad \phi_{4,t}(\infty) = 0$$

and the inverse is given by

$$\phi_{4,t}^{-1}(w) = w + \frac{t}{w}, \quad \phi_{4,t}^{-1}(0) = \infty.$$

For the number of discrete eigenvalues of L it follows:

Corollary 5.5 *Let $p > 0$, L_0 a bounded operator with $\sigma(L_0) = [a, b] \subseteq \mathbb{R}$ and $\|R_{L_0}(\lambda)\| \leq \frac{M}{\mathrm{dist}(\lambda,[a,b])}$ for all $\lambda \in \rho(L_0)$ with fixed $M \geq 1$.*

(a) Let $0 < \gamma < 1$.

(i) If N is chosen such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda\in\mathcal{E}_{s^\gamma}} \|R_{L_0}(\lambda)\|}$ then

$$\mathcal{N}_L(\mathcal{E}_s) \leq \frac{2^p M^p C_p s^{\gamma(2p-1)+1}}{(s^{1-\gamma}-1)\left((s^{2\gamma}-1)(s^\gamma-1) - 2Ms^{2\gamma}\alpha_{N+1}(K)\right)^p} \times \sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p,$$

(ii) if $K \in \mathcal{S}_p(X)$ then

$$\mathcal{N}_L(\mathcal{E}_s) \leq \frac{2^p M^p C_p s^{\gamma(2p-1)+1}}{(s^{1-\gamma}-1)\left((s^{2\gamma}-1)(s^\gamma-1)\right)^p} \|K\|_{\mathcal{S}_p}$$

for all $s > 1$.

(b) (i) If N is such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda\in\tilde{\mathcal{E}}_s} \|R_{L_0}(\lambda)\|}$ then

$$\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right) \leq \frac{C_p M^p s^{1+\frac{p}{2}}}{(s-1)\left((s-1)^{\frac{3}{2}} - Ms^{\frac{1}{2}}\alpha_{N+1}(K)\right)^p} \times \sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p,$$

(ii) if $K \in \mathcal{S}_p(X)$ then

$$\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right) \leq \frac{C_p M^p s^{1+\frac{p}{2}}}{(s-1)^{\frac{3}{2}p+1}} \|K\|_{\mathcal{S}_p}^p$$

for all $s > 1$.

Proof: Due to Theorem 5.1 there is the following bound for the number of eigenvalues in $\mathcal{E}_s$ for $s > 1$:
If N is such that $\alpha_{N+1}(K) < \frac{1}{\sup_{\lambda\in\mathcal{E}_s} \|R_{L_0}(\lambda)\|}$ then

$$\mathcal{N}_L(\mathcal{E}_s) \leq \frac{C_p \sup_{z\in\mathcal{E}_t} \|R_{L_0}(\lambda)\|^p}{\log \frac{1}{r_{\mathcal{E}_t}(\mathcal{E}_s)} \left(1 - \alpha_{N+1}(K) \sup_{z\in\mathcal{E}_t} \|R_{L_0}(\lambda)\|\right)^p} \sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p \tag{5.13}$$

for $1 < t < s$.
Hence, we have to compute

$$r_{\mathcal{E}_t}(\mathcal{E}_s) := \sup_{z\in\mathcal{E}_s} |\phi_{2,t}(z)|$$

and to estimate $\|R_{L_0}(z)\|$ on $\mathcal{E}_t$ in terms of t.
To compute the quantity $r_{\mathcal{E}_t}(\mathcal{E}_s)$ note, that $\phi_{2,t}(\infty) = 0$ and therefore, due to the maximum principle of holomorphic functions (see e.g. [1] p. 134)

$$\begin{aligned}r_{\mathcal{E}_t}(\mathcal{E}_s) &= \sup_{z\in\partial\mathcal{E}_s} |\phi_{2,t}(z)| \\ &= \sup_{\theta\in[-\pi,\pi]} \left| t\frac{\frac{1}{s}e^{i\theta} - se^{-i\theta} + \sqrt{\left(\frac{1}{s}e^{i\theta} - se^{-i\theta}\right)^2 - 4}}{2}\right| = \frac{t}{s}.\end{aligned}$$

Since every $\lambda \in \mathcal{E}_t$ can be rewritten as $\lambda = w + w^{-1}$ where $w = \frac{1}{r}e^{i\theta}$ with $r > t > 1$ and $\theta \in [-\pi, \pi]$ a bound for the resolvent is given by

$$\begin{aligned}\sup_{r>t,\theta\in[-\pi,\pi]} \|R_{L_0}(\lambda)\| &\leq \sup_{r>t,\theta\in[-\pi,\pi]} \frac{M}{\operatorname{dist}(\lambda, [-2, 2])} \\ &\overset{\text{Theorem 4.5}}{\leq} M2 \sup_{r>t,\theta\in[-\pi,\pi]} \frac{\left|\frac{1}{r}e^{i\theta}\right|}{\left|(\frac{1}{r}e^{i\theta})^2 - 1\right| \left|1 - \frac{1}{r}e^{i\theta}\right|} \\ &= 2M\frac{t^2}{(t^2 - 1)(t - 1)}.\end{aligned}$$

On account of (5.13) a possible bound for the number of eigenvalues in $\mathcal{E}_s$ is given by

$$\begin{aligned}\mathcal{N}_L(\mathcal{E}_s) &\leq \frac{2^pM^pC_p}{\log\frac{1}{\frac{t}{s}}\left(\frac{(t^2-1)(t-1)}{t^2} - 2M\alpha_{N+1}(K)\right)^p}\sum_{j=1}^{N}(\alpha_{N+1}(K) + \alpha_j(K))^p \\ &\leq \frac{2^pM^pC_p}{\left(1-\frac{t}{s}\right)\left(\frac{(t^2-1)(t-1)}{t^2} - 2M\alpha_{N+1}(K)\right)^p}\sum_{j=1}^{N}(\alpha_{N+1}(K) + \alpha_j(K))^p .\end{aligned} \tag{5.14}$$

Note, that (5.14) is true for every t with $1 < t < s$. Hence, it remains true for $t := s^\gamma$ for any $0 < \gamma < 1$:

$$\begin{aligned}\mathcal{N}_L(\mathcal{E}_s) \leq& \frac{2^pM^pC_ps^{1+\gamma(2p-1)}}{(s^{1-\gamma} - 1)\left((s^{2\gamma} - 1)(s^\gamma - 1) - s^{2\gamma}2M\alpha_{N+1}(K)\right)^p} \\ &\times \sum_{j=1}^{N}(\alpha_{N+1}(K) + \alpha_j(K))^p .\end{aligned}$$

This proves (a) (i). Taking the limit $N \to \infty$ and the same limiting arguments as in the proof of Theorem 5.1 shows the validity of (a) (ii).
In a similar manner one can show the validity of (b). Since $\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right) \subseteq \tilde{\mathcal{E}}_s$ (for illustration see also Figure 5.2), Theorem 5.1 gives the bound

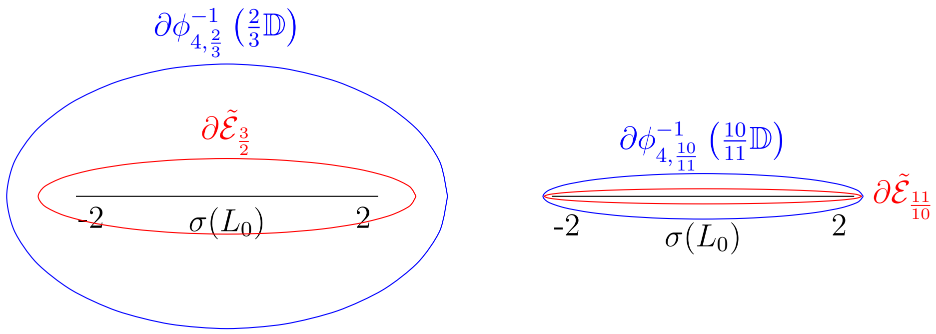

Figure 5.2: The boundary of the sets $\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)$ (blue) and $\tilde{\mathcal{E}}_s$ (red) for $s = \frac{3}{2}$ (left) and $s = \frac{11}{10}$ (right).

$$\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right) \leq \frac{C_p \sup_{z\in\tilde{\mathcal{E}}_s}\|R_{L_0}(z)\|}{\log\frac{1}{r_{\tilde{\mathcal{E}}_s}\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right)}\left(1-\alpha_{N+1}(K)\sup_{z\in\tilde{\mathcal{E}}_s}\|R_{L_0}(z)\|\right)^p} \times \sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p . \tag{5.15}$$

Then

$$r_{\tilde{\mathcal{E}}_s}\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right) = \sup_{w\in\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)}|\phi_{4,s}(w)| = \frac{1}{s} \tag{5.16}$$

and via the estimates in Theorem 4.6

$$\sup_{\lambda\in\tilde{\mathcal{E}}_s}\|R_{L_0}(\lambda)\| \leq \sup_{\lambda\in\tilde{\mathcal{E}}_s}\frac{M}{\operatorname{dist}(\lambda,[-2,2])} \leq \frac{Ms^{\frac{1}{2}}}{(s-1)^{\frac{3}{2}}} \tag{5.17}$$

Combining (5.15), (5.16), (5.17) and using inequality $\log(s) \geq \frac{s-1}{s}$ we have

$$\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right) \leq \frac{C_p M^p s^{1+\frac{p}{2}}}{(s-1)\left((s-1)^{\frac{3}{2}} - Ms^{\frac{1}{2}}\alpha_{N+1}(K)\right)^p} \tag{5.18}$$

$$\times\sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p \tag{5.19}$$

which proves (b) (i). Following the arguments of the proof of (a) one can derive from (5.19) the estimate in (b) (ii). $\square$

5.1.4 Lieb-Thirring-type inequalities

Now we can extend our results in terms of bounds on the moments of eigenvalues of L, if $K \in \mathcal{S}_p(X)$.

Corollary 5.6 *Let $p > 0$ and $K \in \mathcal{S}_p(X)$.*

(a) If $q > p + 1$

$$\sum_{\lambda \in \sigma_d(L), |\lambda| > \|L_0\|} (|\lambda| - \|L_0\|)^q < \infty.$$

(b) If $\sigma(L_0) = [a, b]$ and $\|R_{L_0}(\lambda)\| \le \frac{M}{\mathrm{dist}(\lambda,[a,b])}$ for fixed $M > 0$ then

(i)

$$\sum_{\lambda \in \sigma_d(L)} \mathrm{dist}(\lambda, [a, b])^q < \infty \text{ for each } q > \frac{3}{2}p + 1,$$

(ii)

$$\sum_{\lambda \in \sigma_d(L), |\mathrm{Re}(\lambda)| \ge 2} \mathrm{dist}(\lambda, [a, b])^q < \infty \text{ for each } q > p + \frac{1}{2}.$$

Moreover, each of the previous sums is not only finite but they can also be bounded by constants which only depend on $p, q, \|L_0\|$ and $\|K\|_{\mathcal{S}_p}$.

Proof: Proof of (a):
We have, using integration by parts and Corollary 5.3,

$$\sum_{\lambda \in \sigma_d(L), |\lambda| > \|L_0\|} (|\lambda| - \|L_0\|)^q = q \int_{\|L_0\|}^{\infty} \mathcal{N}_L\left((s\mathbb{D})^c\right)(s - \|L_0\|)^{q-1} ds \tag{5.20}$$

$$\le \frac{C_p(p+1)^{p+1}}{p^p} \|K\|_{\mathcal{S}_p}^p \int_{\|L_0\|}^{\|L\|+\|K\|} \frac{s}{(s - \|L_0\|)^{p+2-q}} ds. \tag{5.21}$$

The right hand side of (5.21) is finite if and only if $q > p + 1$. Proof of (b): Without loss of generality $[a, b]$ is assumed to be the intervall $[-2, 2]$.

Proof of (b)(i):
Let $\lambda \in \partial\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right) = \phi_{4,s}^{-1}\left(\partial\frac{1}{s}\mathbb{D}\right)$ then by Koebe's distortion theorem (Theorem 4.7)

$$\begin{aligned}\operatorname{dist}(\lambda, \partial\tilde{\mathcal{E}}_s) \leq& 2|(\phi_{4,s}^{-1})'(\phi_{4,s}(\lambda))(1-|\phi_{4,s}(\lambda)|)| = \left|1 - \frac{s}{\phi_{4,s}^2(\lambda)}\right|(1-|\phi_{4,s}(\lambda)|)\\ \overset{|\phi_{4,s}(\lambda)|=\frac{1}{s}}{\leq}& \frac{1+s^3}{s}(s-1)\end{aligned}$$

and due to Theorem 4.6

$$\begin{aligned}\operatorname{dist}(\lambda, [-2,2]) \leq& \operatorname{dist}(\lambda, \partial\tilde{\mathcal{E}}_s) + \operatorname{dist}(\partial\tilde{\mathcal{E}}_s, [-2,2])\\ \leq& \frac{1+s^3}{s}(s-1) + (s-1) = (s-1)\frac{1+s+s^3}{s}.\end{aligned}$$

Then (once again using integration by parts)

$$\begin{aligned}\sum_{\lambda\in\sigma_d(L)} \operatorname{dist}(\lambda,[-2,2])^q =& \sum_{\substack{\lambda\in\sigma_s(L)\cap\partial\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\\ s>1}} \operatorname{dist}(\lambda,[-2,2])^q\\ \leq& \sum_{\substack{\lambda\in\sigma_s(L)\cap\partial\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\\ s>1}} M_1(s-1)^q\\ =& q\int_1^\infty M_1\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right)(s-1)^{q-1}ds\\ \overset{(*)}{=}& q\int_1^{\|L_0\|+\|K\|+1} M_1\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right)(s-1)^{q-1}ds\\ \overset{\text{Corollary 5.5 b(ii)}}{\leq}& \int_1^{\|L_0\|+\|K\|+1} M_2\frac{(s-1)^{q-1}}{(s-1)^{\frac{3}{2}p+1}}ds\\ =& \int_1^{\|L_0\|+\|K\|+1} \frac{M_2}{(s-1)^{\frac{3}{2}p-q+2}}ds \qquad (5.22)\end{aligned}$$

with the finite constants $M_1 := \sup_{s\in[1,\|L_0\|+\|K\|+1]} \frac{1+s+s^3}{s}$ and $M_2 := q\cdot M_1\cdot M^p\cdot C_p\cdot\|K\|_{\mathcal{S}_p}^p\cdot\sup_{s\in[1,\|L_0\|+\|K\|+1]} s^{1+\frac{p}{2}}$.
(5.22) is finite if and only if $\frac{3}{2}p - q + 2 < 1 \Leftrightarrow q > \frac{3}{2}p + 1$.
Note, that (*) is possible, since $\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right) = 0$ for all $s > \|L_0\|+\|K\|+1$.
In fact, if $s > \|L_0\| + \|K\| + 1$ and $\lambda \in \phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)$ then there are $r > s$ and $\theta \in [-\pi,\pi]$ with

$$\lambda = \phi_{4,s}^{-1}\left(\frac{1}{r}e^{i\theta}\right) = \frac{1}{r}e^{i\theta} + rse^{-i\theta}.$$

Thus

$$|\lambda| \geq rs - \frac{1}{r} > (\|L_0\| + \|K\| + 1)^2 - \frac{1}{\|L_0\| + \|K\| + 1} > \|L_0\| + \|K\| \geq \|L\|$$

and therefore $\lambda \notin \sigma_d(L)$.
Proof of (b) (ii):
By symmetrie it is sufficient to show the finiteness for all eigenvalues with $\mathrm{Re}(\lambda) \geq 2$.
Remeber, if $\lambda \in \partial\mathcal{E}_s$ then $\lambda = \frac{e^{i\theta}}{s} + \frac{s}{e^{i\theta}} = \left(\frac{1}{s} + s\right)\cos(\theta) + i\left(\frac{1}{s} - s\right)\sin(\theta)$ with $\theta \in [-\pi, \pi]$. Additionally, if $\mathrm{Re}(\lambda) \geq 2$ (i.e. $1 \leq \cos(\theta) \leq \frac{2s}{1+s^2}$) then

$$\begin{aligned}
\mathrm{dist}(\lambda, [-2,2]) =& |\lambda - 2| = \left| \frac{\left(\frac{e^{i\theta}}{s} - 1\right)^2}{\frac{e^{i\theta}}{s}} \right| \\
=& s\left(\left(\frac{1}{s}\cos(\theta) - 1\right)^2 + \left(\frac{1}{s}\sin(\theta)\right)^2\right) \\
=& s\left(\left(\frac{1}{s}\cos(\theta) - 1\right)^2 + \frac{1}{s^2}\left(1 - \cos^2(\theta)\right)\right).
\end{aligned}$$

A short discussion of the function $f(x) := \left(\frac{1}{s}x - 1\right)^2 + \frac{1}{s^2}(1 - x^2)$ shows that f is monotonically decreasing on $[1, \frac{2s}{1+s^2}]$ and therefore f has to take its maximum in $x = 1$ with

$$f\left(\frac{2s}{1+s^2}\right) = \frac{(1-s)^2}{s}.$$

Hence

$$\mathrm{dist}(\lambda, [-2,2]) \leq \frac{(1-s)^2}{s} \leq (1-s)^2 \text{ for all } \lambda \in \mathcal{E}_s \text{ with } s > 1 \text{ and } \mathrm{Re}(\lambda) \geq 2.$$

With the same argumentation as above we have

$$\begin{aligned}
\sum_{\substack{\lambda\in\sigma_d(L)\\ \mathrm{Re}(\lambda)\geq 2}} \operatorname{dist}(\lambda,[-2,2])^q =& \sum_{\substack{\lambda\in\sigma_d(L)\cap\partial\mathcal{E}_s\\ \mathrm{Re}(\lambda)\geq 2, s>1}} \operatorname{dist}(\lambda,[-2,2])^q \\
\leq& \sum_{\substack{\lambda\in\sigma_d(L)\cap\partial\mathcal{E}_s\\ \mathrm{Re}(\lambda)\geq 2, s>1}} (s-1)^{2q} \\
\leq& 2q\int_1^\infty \mathcal{N}_L(\mathcal{E}_s)(s-1)^{2q-1}ds \\
\overset{(**)}{=}& 2q\int_1^{\|L_0\|+\|K\|+1} \mathcal{N}_L(\mathcal{E}_s)(s-1)^{2q-1}ds \\
\overset{\substack{\text{Corollary 5.5(a)(ii)}\\ \gamma=\frac{1}{2}}}{\leq}& M_3\int_1^{\|L_0\|+\|K\|+1} \frac{(s-1)^{2q-1}}{(s-1)^p(s^{\frac{1}{2}}-1)^{p+1}}ds \\
\leq& M_3\int_1^{\|L_0\|+\|K\|+1} \frac{(s-1)^{2q-1}(s^{\frac{1}{2}}+1)^{p+1}}{(s-1)^p(s-1)^{p+1}}ds \\
\leq& M_4\int_1^{\|L_0\|+\|K\|+1} \frac{1}{(s-1)^{2p-2q+2}}ds \qquad (5.23)
\end{aligned}$$

with $M_3 := \sup_{s\in[1,\|L_0\|+\|K\|+1]} 2q\cdot 2^p\cdot M^p\cdot C_p\cdot \|K\|_{\mathcal{S}_p}^p\cdot s^{p+\frac{1}{2}}$ and $M_4 := \sup_{s\in[1,\|L_0\|+\|K\|+1]} M_3\cdot(s^{\frac{1}{2}}+1)^{p+1}$.
(5.23) is finite if and only if $2p-2q+2<1 \Leftrightarrow q>p+\frac{1}{2}$.
Using similar arguments as in the proof of b(i), one can see that $\mathcal{N}_L(\mathcal{E}_s)=0$ for all $s>\|L_0\|+\|K\|+1$, and therefore equality (**) is true. □

Remark 5.7 (a) Note that the advantage of the first case, where we estimated the number of eigenvalues in the complement of balls with radius bigger than $\|L_0\|$, is that it is applicable for all $L=L_0+K$. The disadvantage is, that it may happen that $\sigma(L_0)\cap\{\lambda : |\lambda|\geq\|L_0\|\}=\emptyset$. Then

$$\#\sigma_d(L)\cap\{\lambda : |\lambda|\geq\|L_0\|\}<\infty$$

and therefore

$$\sum_{\lambda\in\sigma_d(L),|\lambda|>\|L_0\|} (|\lambda|-\|L_0\|)^p<\infty$$

is trivial.
The disadvantage of the second case (eigenvalues in the unbounded

component of ellipses) is, that there are the restrictions on the spectrum and on the behaviour of the resolvent norm of L_0 ($\sigma(L_0) = [a,b]$, $\|R_{L_0}(\lambda)\| \leq M\mathrm{dist}(\lambda,[a,b])^{-1}$). But the advantage is, that all discrete eigenvalues in the Lieb-Thirring-type inequalities are counted and therefore the result is not trivial.

(b) Also in the case $\sigma(L_0) = [-2,2]$ and $\|R_{L_0}(\lambda)\| \leq \frac{M}{\mathrm{dist}(\lambda,[-2,2])}$ it may happen that $\sup\{|\lambda| : \lambda \in \sigma(L_0)\} = \|L_0\| = 2$. Then both, estimating the number of eigenvalues in the complement of a closed disk with radius bigger than 2 and estimating the eigenvalues in complements of ellipses are non trivial.
Nevertheless, the second method (eigenvalues in complements of ellipses) provides stronger results.
Although with the first method we can go arbitrarily close to the spectrum of L_0 (i.e. we come arbitrarily close to the points -2 and 2), there is no possibility to come close to points in (-2,2).
Method 2 allows to get arbitrary close to every point of the intervall [-2,2] (see also Figure 5.3).

However, both methods are applicable to treat eigenvalues of L with

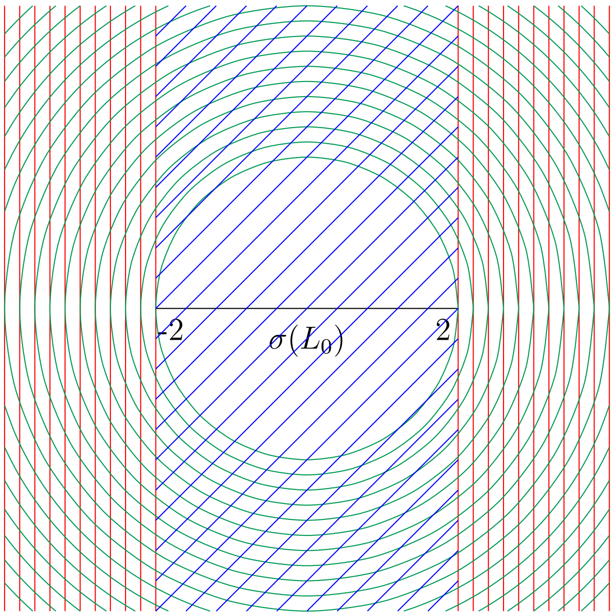

Figure 5.3: $(\|L_0\|\overline{\mathbb{D}})^c$ (green), $\{z \in \mathbb{C}\,, |\mathrm{Re}(z)| \geq 2\}$ (red), $\{z \in \mathbb{C}\,, |\mathrm{Re}(z)| \geq 2\}$ (blue).

real part bigger than 2.
Where the first method yields

$$\sum_{\substack{\lambda\in\sigma_d(L)\\|\mathrm{Re}(\lambda)|>2}} (|\lambda| - \|L_0\|)^q = \sum_{\substack{\lambda\in\sigma_d(L)\\|\mathrm{Re}(\lambda)|>2}} \mathrm{dist}(\lambda,[-2,2])^q < \infty \text{ for } q > p+1$$

the second method provides

$$\sum_{\substack{\lambda\in\sigma_d(L)\\|\mathrm{Re}(\lambda)|>2}} \mathrm{dist}(\lambda,[-2,2])^q = \sum_{\substack{\lambda\in\sigma_d(L)\\|\mathrm{Re}(\lambda)|>2}} (|\lambda| - \|L_0\|)^q < \infty \text{ for } q > p+\frac{1}{2}$$

which is stronger.

(c) If one would use $\mathcal{N}_L(\mathcal{E}_s)$, instead of $\mathcal{N}_L\left(\phi_{4,s}^{-1}\left(\frac{1}{s}\mathbb{D}\right)\right)$ (as it was done in Corollary 5.6 (b) (i)), to get information on the moments of eigenvalues, one would obtain quite different results, i.e.

$$\sum_{\lambda\in\sigma_d(L)} \mathrm{dist}(\lambda,[-2,2])^q < \infty$$

with $q > 2p+1$, instead of $q > \frac{3}{2}p+1$. The reason, is the different asymptotic behaviour of $\|R_{L_0}(\lambda)\|$ on $\mathcal{E}_s$ as $s \to 1$. In fact, in this case ($\lambda \in \partial\mathcal{E}_s$)

$$\|R_{L_0}(\lambda)\| \leq \frac{M}{\mathrm{dist}(\lambda,[-2,2])} = O\left(\frac{1}{(s-1)^2}\right), \text{ as } s \to 1,$$

in contrast to

$$\|R_{L_0}(\lambda)\| \leq \frac{M}{\mathrm{dist}(\lambda,[-2,2])} = O\left(\frac{1}{s-1}\right), \text{ as } s \to 1,$$

for $\lambda \in \partial\tilde{\mathcal{E}}_s$ (see Figure 5.4).

5.2 Possible accumulation points of the discrete spectrum

It was already mentioned, that the discrete spectrum of an operator L can only accumulate in the essential spectrum of L.

In this section two criterions, which exclude special points or subsets of $\sigma_e(L)$ to be accumulation points of the discrete spectrum, are presented. Both criterions use the behaviour of the operator valued map $\lambda \mapsto K_{L_0}(\lambda)$. While the first criterion (Section 5.2.1) deals with a continuation argument the second criterion (Section 5.2.2) uses a boundedness argument.

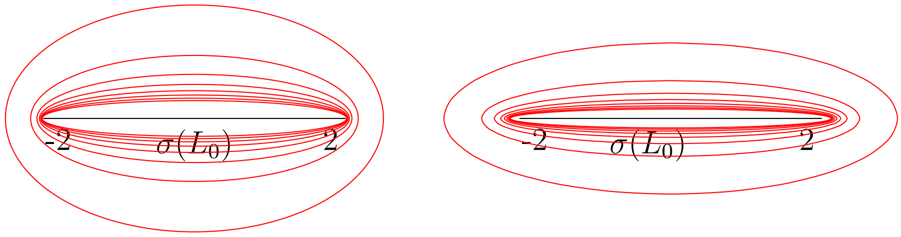

Figure 5.4: On the left $\partial\mathcal{E}_s$, on the right $\partial\tilde{\mathcal{E}}_s$ both for $s = 1 + \frac{1}{n}$ with $n = 1, ..., 8$. Note the different asymptotic behaviour as $s \to 1$.

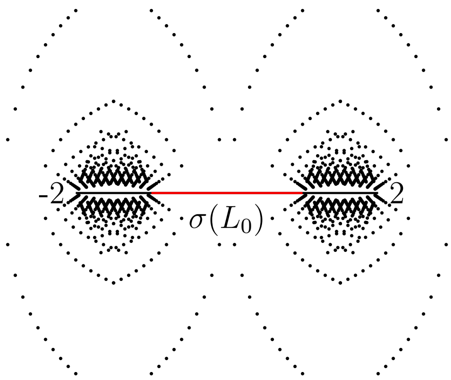

Figure 5.5: This graphic illustrates what may happen. Here $\sigma(L_0) = [-2, 2]$. The discrete eigenvalues do not accumulate at the red intervall.

5.2.1 A continuity criterion

Throughout this subsection we make the following assumption.

Assumption 5.8 L_0 and K are operators on X such that

(a) the spectrum of L_0 is real, connected and purely essential, i.e. $\sigma(L_0) = \sigma_e(L_0) = [a, b] \subseteq \mathbb{R}$,

(b) K belongs to a quasi-Banach ideal $(\mathfrak{B}(X), \|\cdot\|_{\mathfrak{B}}, c_{\mathfrak{B}})$ with quasi-Banach constant $c_{\mathfrak{B}} \geq 1$, which admits a determinant $\tilde{\det}$ with the following properties:

  (i) $\tilde{\det}(\mathbb{1} - K)$ is well defined for all $K \in \mathfrak{B}(X)$,

(ii) $\tilde{\det}(\mathbb{1} - K) = 0$ if and only if $1 \in \sigma(K)$ for all $K \in \mathfrak{B}(X)$,

(iii) for all $\mathfrak{B}(X)$-valued maps $K(\cdot)$ which are analytic on a domain $G \subseteq \mathbb{C}$ the function $\tilde{\det}(\mathbb{1} - K(\cdot))$ is holomorphic on G,

(iv) there is a monotone increasing function g defined on $[0, \infty)$ with

$$|\tilde{\det}(\mathbb{1} - K)| \leq g(\|K\|_{\mathfrak{B}}) \text{ for all } K \in \mathfrak{B}(X).$$

We want to prove a criterion which excludes subsets or points in $[a, b]$ to be accumulation points of the discrete spectrum. Before we will do this we have to modify a theorem due to Seiler-Simon, (see e.g. [15] Theorem II.4.1) which is stated for normed spaces, to quasi-normed spaces.

Theorem 5.9 *Let f be a complex-valued function defined on a complex quasi-normed space $\mathfrak{N}$ with quasi-constant $c_{\mathfrak{N}}$. Suppose*

(a) the function $\lambda \mapsto f(A + \lambda B)$ is an entire function for all $A, B \in \mathfrak{N}$ and

(b) there is a monotone increasing function G on $[0, \infty)$ such that

$$|f(A)| \leq G(\|A\|_{\mathfrak{N}}) \text{ for all } A \in \mathfrak{N}.$$

Then for all $A, B \in \mathfrak{N}$,

$$|f(A) - f(B)| \leq \|A - B\|_{\mathfrak{N}} G\left(c_{\mathfrak{N}}^2(\|A\|_{\mathfrak{N}} + \|B\|_{\mathfrak{N}}) + c_{\mathfrak{N}}\right).$$

Proof: For $A = B$ this assertion is trivial. If $A \neq B$ we set

$$h(\lambda) := f\left(\frac{1}{2}(A + B) + \lambda(A - B)\right).$$

Then by assumption h is an entire function and

$$|f(A) - f(B)| = \left|h\left(\frac{1}{2}\right) - h\left(-\frac{1}{2}\right)\right| \leq \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} |h'(t)|$$

From Cauchy's integral formula we get for any $\rho > 0$

$$\sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} |h'(t)| = \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} \frac{1}{2\pi} \left|\int_{\partial(\rho\mathbb{D})} \frac{h(\zeta + t)}{\zeta^2} d\zeta\right| \leq \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} \frac{1}{\rho} \left(\sup_{|\zeta| = \rho} |h(\zeta + t)|\right)$$
$$\leq \frac{1}{\rho} \sup_{|\lambda| \leq \rho + \frac{1}{2}} |h(\lambda)|.$$

Hence, for all $\rho > 0$

$$|f(A) - f(B)| \le \frac{1}{\rho} \sup_{|\lambda| \le \rho + \frac{1}{2}} |h(\lambda)|. \tag{5.24}$$

This inequality holds in particular for $\rho = \|A - B\|_{\mathfrak{N}}^{-1}$. In this case for all $|\lambda| \le \rho + \frac{1}{2}$ we have

$$\begin{aligned} \left\| \frac{1}{2}(A+B) + \lambda(A-B) \right\|_{\mathfrak{N}} &\le c_{\mathfrak{N}} \frac{1}{2} \left(\|A+B\|_{\mathfrak{N}} + \|A-B\|_{\mathfrak{N}} \right) + c_{\mathfrak{N}} \rho \|A-B\|_{\mathfrak{N}} \\ &\le c_{\mathfrak{N}}^2 (\|A\|_{\mathfrak{N}} + \|B\|_{\mathfrak{N}}) + c_{\mathfrak{N}}, \end{aligned}$$

where $c_{\mathfrak{N}} \ge 1$ denotes the constant in the quasi-triangle inequality.
Therefore, this and assumption (ii) in the formulation of this theorem imply for $|\lambda| \le \rho + \frac{1}{2}$

$$\begin{aligned} |h(\lambda)| = \left| f\left(\frac{1}{2}(A+B) + \lambda(A-B) \right) \right| &\le G\left(\left\| \frac{1}{2}(A+B) + \lambda(A-B) \right\|_{\mathfrak{N}} \right) \\ &\le G\left(c_{\mathfrak{N}}^2 (\|A\|_{\mathfrak{N}} + \|B\|_{\mathfrak{N}}) + c_{\mathfrak{N}} \right). \end{aligned}$$

Thus ((5.24) with $\rho = \|A_B\|_{\mathfrak{N}}^{-1}$)

$$|f(A) - f(B)| \le \|A - B\|_{\mathfrak{N}} G\left(c_{\mathfrak{N}}^2 (\|A\|_{\mathfrak{N}} + \|B\|_{\mathfrak{N}}) + c_{\mathfrak{N}} \right).$$

□

Theorem 5.10 *Let $L = L_0 + K$ satisfy Assumption 5.8 and let $E \subsetneq [a,b]$ be an open set (open according to $\mathbb{R}$). If the operator valued map $\rho(L_0) \ni \lambda \mapsto KR_{L_0}(\lambda)$ can be continuously extended to $E \cup \rho(L_0)$ (with respect to $\|\cdot\|_{\mathfrak{B}}$) then $E \cap \overline{\sigma_d(L)} = \emptyset$.*
If $\lambda_0 \in [a,b]$ is a single point with the property that there is a continuous extension of $KR_{L_0}(\cdot)$ (with respect to $\|\cdot\|_{\mathfrak{B}}$), let us call this continuation K_{λ_0}, and 1 is not a discrete eigenvalue of K_{λ_0}, then λ_0 is not an accumulation point of $\sigma_d(L)$.

Proof: Under Assumption 5.8 we know that $\lambda \mapsto \tilde{\det}\,(\mathbb{1} - (A + \lambda B))$ is an entire function for all $A, B \in \mathfrak{B}(X)$, since $\lambda \mapsto A + \lambda B$ is analytic and $\mathfrak{B}(X)$-valued on the entire complex plane $\mathbb{C}$. Using Theorem 5.9 we see that $\tilde{\det}$ is locally Lipschitz-continuous[1] (with respect to $\|\cdot\|_{\mathfrak{B}}$), i.e.

$$\begin{aligned} &|\tilde{\det}(\mathbb{1} - A) - \tilde{\det}(\mathbb{1} - B)| && (5.25) \\ \le &\|A - B\|_{\mathfrak{B}}\, g\left(c_{\mathfrak{B}} (\|A\|_{\mathfrak{B}} + \|B\|_{\mathfrak{B}}) + c_{\mathfrak{B}} \right) \text{ for all } A, B \in \mathfrak{B}(X). && (5.26) \end{aligned}$$

[1] $c_{\mathfrak{B}}$ denotes the constant in the quasi-triangle inequality according to $\mathfrak{B}(X)$.

Let $d(\lambda) := \tilde{\det}\,(\mathbb{1} - KR_{L_0}(\lambda))$ be defined on $\rho(L_0)$, and let $\lambda_0 \in \sigma_e(L_0)$ be any point with the property, that there exists an operator $K_{\lambda_0} \in \mathfrak{B}(X)$ with

$$\|KR_{Z_0}(\lambda) - K_{\lambda_0}\|_{\mathfrak{B}} \stackrel{\lambda\to\lambda_0}{\longrightarrow} 0, \tag{5.27}$$

then due to (5.26), we can derive

$$\begin{aligned} |d(\lambda) - \tilde{\det}\,(\mathbb{1} - K_{\lambda_0})\,| \leq & \|KR_{L_0}(\lambda) - K_{\lambda_0}\|_{\mathfrak{B}} \\ & \times g\,(c_{\mathfrak{B}}(\|KR_{L_0}(\lambda)\|_{\mathfrak{B}} + K_{\lambda_0}\|_{\mathfrak{B}}) + c_{\mathfrak{B}})\,, \end{aligned}$$

and obtain, that d can be extended continuously to λ_0.
Now, if the set of points in E, for which d is continuously extendable, is an open set (open according to $\mathbb{R}$) the Theorem of Morera (see e.g. [32]) tells us that even d is holomorphically extendable to $\rho(L_0) \cup E$. Since the zeros of every non-zero function do not accumulate in its domain, we know that the zeros of d cannot accumulate in E. But this is equivalent to the assertion, that the discrete spectrum of L does not accumulate to E.
If there is only a single point $\lambda_0 \in \sigma_{ess}(Z_0)$, with the property that there is a continuous extension K_{λ_0} of $KR_{L_0}(\cdot)$ in λ_0, we obtain that there is also a continuous extension of d in λ_0 realized by $\tilde{\det}(\mathbb{1} - K_{\lambda_0})$. If $1 \notin \sigma(K_{\lambda_0})$, then $\tilde{\det}(\mathbb{1} - K_{\lambda_0}) \neq 0$. Hence it is not possible for the zeros of d to accumulate at λ_0 and also not for the discrete spectrum of L. □

Remark 5.11 For a slightly stronger result have a look at Theorem 10.1 in the appendix.

5.2.2 A boundedness criterion

The previous subsection used strong assumptions on L_0 and K. Now we only assume that L_0 is a bounded operator with purely essential spectrum and K a compact operator.

Theorem 5.12 *Let $L = L_0 + K$. Assume that $\Omega \subseteq \rho(L_0)$ is a subset of the unbounded component of $\rho(L_0)$. If*

$$\|KR_{L_0}(\lambda)\| < 1 \text{ for all } \lambda \in \Omega \tag{5.28}$$

then the discrete eigenvalues of L in the unbounded component of $\rho(L_0)$ cannot accumulate at $\overline{\Omega} \cap \sigma_e(L_0)$.

Proof: Remember that λ is a discrete eigenvalue of $L = L_0 + K$ if and only if $1 \in KR_{L_0}(\lambda)$ which implies $\|KR_{L_0}(\lambda)\| \geq 1$. □

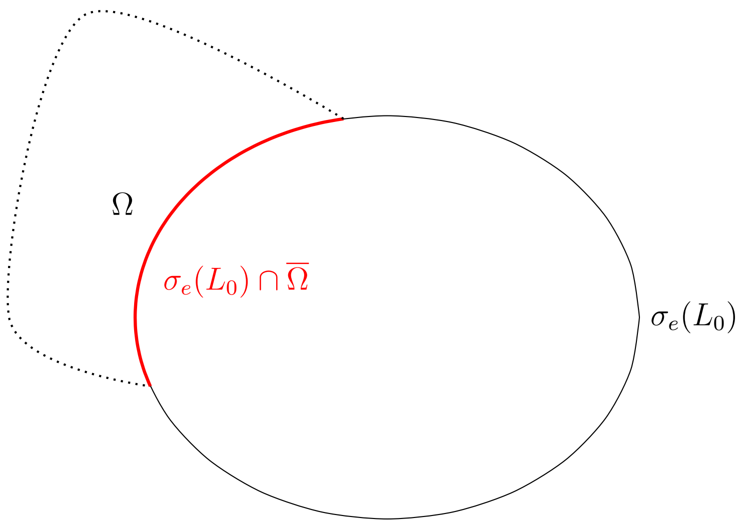

Figure 5.6: This figure illustrates the assertion of Theorem 5.12. If $\|KR_{L_0}(\lambda)\| < 1$ for all $\lambda \in \Omega$, then there are no discrete eigenvalues of L in Ω. Therefore $\overline{\Omega} \cap \sigma_e(L_0)$ fails to be a set of accumulationpoints of the discrete spectrum in Ω.

Chapter 6

Eigenvalues in the bounded component of the resolvent set of the unperturbed operator

In the previous chapters we were focused on the discrete eigenvalues of the operator $L = L_0 + K$ in the unbounded component of $\rho(L_0)$. In fact, if Ω is a subset of the unbounded component of $\rho(L_0)$, we know that

$$\Omega \cap \sigma(L) \subseteq \sigma_d(L), \tag{6.1}$$

i.e. all spectral points of L in Ω are discrete eigenvalues (Proposition 1.8). However, if Ω is a non empty and bounded component of $\rho(L_0)$, inclusion (6.1) has not to be true.
In fact, in this case it may happen that

$$\Omega \cap \sigma(L) = \Omega,$$

even if K is a rank one operator (see Example 6.1).
Nevertheless, it seems to be natural to ask whether it is possible to study the eigenvalues of L in bounded components of $\rho(L_0)$ with the methods introduced in the previous chapters. That is, if $K \in \mathfrak{A}(X)$[1], to identify the discrete eigenvalues of L in Ω with the zeros of the function

$$d(\lambda) := \det_{\mathfrak{A}} (\mathbb{1} - KR_{L_0}(\lambda)) \text{ for } \lambda \in \Omega, \tag{6.2}$$

The function in (6.2) is well defined, holomorphic and $d(\lambda) = 0$ if and only if λ is an eigenvalue of Ω. Hence, there is no need of restricting to the unbounded component of $\rho(L_0)$. But it may happen, that an eigenvalue of

[1]For the denotation of $\mathfrak{A}(X)$ and $\det_{\mathfrak{A}}$ have a look to Section 3.1.

L in the bounded component of $\rho(L_0)$ fails to be an discrete eigenvalue. In Theorem 6.2 a criterion is proved which allows to distinguish between pure point spectrum and discrete spectrum.

Example 6.1 Let $S : l^1(\mathbb{Z}) \to l^1(\mathbb{Z})$ be the two sided shift operator defined by $(Sf)(n) := f(n-1)$, or equivalently given by the infinite two sided matrix

$$\begin{pmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

according to the canonical standard basis. It is possible to identify the space $l^1(\mathbb{Z})$ with the space $\mathcal{A} := \{\sum_{n=-\infty}^{\infty} a_n z^n : (a_n) \in l^1(\mathbb{Z}), z \in \partial\mathbb{D}\}$, due to the linear, bijective map $\phi : l^1(\mathbb{Z}) \to \mathcal{A}$ defined by

$$\phi((a_n)) \mapsto \sum_{n=-\infty}^{\infty} a_n z^n.$$

Then the action of S to any function $f \in \mathcal{A}$ is

$$(Sf)(z) = zf(z).$$

Therefore, $\lambda \in \mathbb{C}$ belongs to $\rho(S)$ if and only if for every $g \in \mathcal{A}$ there is a unique function $f \in \mathcal{A}$ with

$$((\lambda \mathbb{1} - S)f)(z) = g(z) \text{ for all } z \in \partial\mathbb{D}. \tag{6.3}$$

Equation (6.3) is equivalent to

$$f(z) = \frac{g(z)}{\lambda - z} \text{ for all } z \in \partial\mathbb{D}.$$

The function $z \mapsto \frac{g(z)}{\lambda - z}$ is an element of the space $\mathcal{A}$, for all $g \in \mathcal{A}$, if and only if

$$\lambda - z \neq 0 \text{ for all } z \in \partial\mathbb{D},$$

which implies $\lambda \notin \partial\mathbb{D}$. Hence, $\sigma(S) = \sigma_e(S) = \partial\mathbb{D}$ (see Proposition 1.6). Let K be the rank one operator defined by

$$Kf := -\langle \delta_0, f \rangle \delta_1 \text{ for all } f \in l^1(\mathbb{Z}),$$

where for each n the symbol δ_n denotes the n-th canonical unit vector in $l^1(\mathbb{Z})$. The perturbed operator $L := S + K$, is given by the infinite matrix

$$\begin{pmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & 0 & 0 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix} \leftarrow \text{first line.}$$

or equivalently defined by $L\delta_0 = 0$ and $L\delta_l = \delta_{l+1}$ if $l \neq 0$.
If one defines

$$g := \sum_{j=-\infty}^{0} \lambda^{|j|}\delta_j \in l^1(\mathbb{Z})$$

for any $\lambda \in \mathbb{D}$ and apply L to g one can see

$$Lg = \sum_{j=-\infty}^{0} \lambda^{|j|+1}\delta_j = \lambda \left(\sum_{j=-\infty}^{0} \lambda^{|j|}\delta_j \right) = \lambda g. \tag{6.4}$$

Hence, each $\lambda \in \mathbb{D}$ is an eigenvalue of L, i.e. $\mathbb{D} \subseteq \sigma(L)$.
This result can be confirmed by determining the zeros of a regularized perturbation determinant.
Since, K is a rank one operator also $KR_S(\cdot)$ is rank one valued. That implies $KR_S(\lambda) \in \mathcal{F}\left(l^1(\mathbb{Z}), \mathrm{Ran}(K)\right)$ for all $\lambda \in \rho(S)$, and therefore $\det_1\left(\mathbb{1} - KR_S(\lambda)\right)$ is well-defined for all $\lambda \in \rho(S)$. Moreover, for rank one operators the determinant can be given explicitly (see e.g. [15] I. Theorem 3.2):

$$\begin{aligned}
\det_1\left(\mathbb{1} - KR_S(\lambda)\right) &= 1 + \langle (\lambda - S)^{-1}\delta_1, \delta_0 \rangle \\
&= \begin{cases} 1 + \left\langle \sum_{k=0}^{\infty} \left(\frac{1}{\lambda}\right)^{k+1} (S)^k \delta_1, \delta_0 \right\rangle & \text{if } |\lambda| > 1 = \|S\|, \\ 1 - \left\langle \sum_{k=0}^{\infty} \lambda^k \left(S^{-1}\right)^{k+1} \delta_1, \delta_0 \right\rangle & \text{if } |\lambda| < 1 = \|S^{-1}\| \end{cases} \\
&= \begin{cases} 1 + \sum_{k=0}^{\infty} \left\langle \frac{1}{\lambda^{k+1}}\delta_{1+k}, \delta_0 \right\rangle & \text{if } |\lambda| > 1, \\ 1 - \sum_{k=0}^{\infty} \left\langle \lambda^k \delta_{-k}, \delta_0 \right\rangle & \text{if } |\lambda| < 1, \end{cases} \\
&= \begin{cases} 1 \text{ if } |\lambda| > 1, \\ 0 \text{ if } |\lambda| < 1. \end{cases}
\end{aligned}$$

Hence $d := \det_1(\mathbb{1} - KR_S(\cdot))$ is a holomorphic function defined on $\mathbb{C} \setminus \partial\mathbb{D}$ with $d \equiv 1$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $d \equiv 0$ on $\mathbb{D}$. Hence, the set of eigenvalues of L coincides with $\mathbb{D}$, which confirms the previous considerations.

As already mentioned, in the theorem below, there is a criterion to decide if a spectral point in the bounded component of $\rho(L_0)$ is a discrete eigenvalue or not.

Theorem 6.2 *Let $L := L_0 + K$ with $K \in \mathfrak{A}(X)$ and $\mathfrak{A}(X)$ defined as in the begining of this section. Let Ω be a bounded component of $\rho(L_0)$, then*

(i) $\mathcal{Z}(d|_\Omega) = \Omega \cap \sigma(L) \subseteq \sigma_d(L)$ if and only if $d(\lambda) := \det_{\mathfrak{A}}(\mathbb{1} - KR_{L_0}(\lambda))$ is not constant on Ω.

(ii) $\Omega \cap \sigma(L) = \Omega$ if $d \equiv 0$ on Ω and each element of Ω is an eigenvalue.

Proof: For the sake of completeness we repeat the arguments of the beginning of Section 3.1. Let $\lambda \in \Omega$ be an eigenvalue of $L = L_0 + K$, with the corresponding eigenfunction f. Then

$$Lf = \lambda f \Leftrightarrow L_0 f + Kf = \lambda f \Leftrightarrow Kf = (\lambda\mathbb{1} - L_0)f.$$

With $g := (\lambda\mathbb{1} - L_0)f$ we have

$$KR_{L_0}(\lambda)g = g$$

which implies $1 \in \sigma(KR_{L_0}(\lambda))$ and therefore $d(\lambda) = 0$. This proves assertion (ii) completely.
To prove (i) we assume that d is not constant. This implies there is at most a countable set of zeros which can only accumulate at the boundary of Ω. This zeros are eigenvalues of L. We have to show that these eigenvalues are discrete (i.e. isolated from the spectrum and of finite algebraic multiplicity). According to Proposition 1.8 it suffices to prove that $\Omega \setminus \mathcal{Z}(d|_\Omega) \subseteq \rho(L_0)$.
The operator valued function $\Omega \ni \lambda \mapsto \lambda - L$ is Fredholm valued (since $\Omega \subseteq \rho(L_0) \subseteq \mathbb{C} \setminus \sigma_e(L)$). This implies, since Ω is connected, that the integer-valued function $\lambda \mapsto \operatorname{ind}(\lambda - L)$ is constant on Ω (Remark 1.5). Remember, that

$$\operatorname{ind}(\lambda - L) := \ker(\lambda - L) - \operatorname{coker}(\lambda - L).$$

If A is an invertible operator, then the index is $\operatorname{ind}(A) = 0$ because $\ker(A) = 0$ and $\operatorname{coker}(A) = 0$. Hence, for every $\lambda \in \Omega \subseteq \rho(L_0)$ the index is

$$\operatorname{ind}(\lambda - L_0) = 0.$$

It follows, since K is a compact operator, that $\text{ind}(\lambda - L_0) = \text{ind}(\lambda - L_0 - K) = \text{ind}(\lambda - L) = 0$ (see Theorem 1.2). Hence, if $\lambda_0 \in \Omega$ is no eigenvalue of L then

$$0 = \text{ind}(\lambda_0 - L) = \ker(\lambda_0 - L) - \text{coker}(\lambda_0 - L) = 0 - \text{coker}(\lambda_0 - L),$$

and therefore $\text{coker}(\lambda_0 - L) = 0$. That means $\lambda_0 - L$ is invertible and therefore $\lambda_0 \in \rho(L) \cap \Omega \neq \emptyset$. Due to Proposition 1.8 $\Omega \cap \sigma(L) \subseteq \sigma_d(L)$.

$\square$

Chapter 7

Applications

For illustration we will apply the results from the previous chapters (in particular Chapter 4) to some special operators. Section 7.1 and Section 7.2 are based on the joint work (M. Demuth and F. Hanauska) [5] and on [17].

7.1 The discrete Laplacian

Let Δ_q be the **discrete Laplacian** on $l^q(\mathbb{Z})$, $1 \leq q \leq \infty$, where the case $q = \infty$ has to be emphasized, since $l^\infty(\mathbb{Z})$ is not compatible to $l^2(\mathbb{Z})$. This operator $\Delta_q : l^q(\mathbb{Z}) \to l^q(\mathbb{Z})$ is given by

$$(\Delta_q f)(n) := f(n-1) + f(n+1), \quad f \in l^q(\mathbb{Z}).$$

Δ_q is a bounded operator on $l^q(\mathbb{Z})$, $q \in [0, \infty]$. It can be rewritten as

$$\Delta_q f = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \end{pmatrix} \tag{7.1}$$

for $f \in l^q(\mathbb{Z})$, $f = \begin{pmatrix} \vdots \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \end{pmatrix}$.

$\Delta_q \in \mathcal{L}(l^q(\mathbb{Z}))$ follows by

$$\|\Delta_q f\|_q \leq 2\|f\|_q, \ f \in l^q(\mathbb{Z}).$$

Proposition 7.1 *The resolvent set of Δ_q is $\mathbb{C}\setminus[-2,2]$ (hence $\sigma(\Delta_q) = \sigma_e(\Delta_q) = [-2,2]$) and the resolvent for $z \in \rho(\Delta_q)$ is given by*

$$R_{\Delta_q}(z) = (z\mathbb{1} - \Delta_q)^{-1} := -\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \dots & b_{-1}(z) & b_0(z) & b_1(z) & b_2(z) & \dots \\ \dots & b_{-2}(z) & b_{-1}(z) & b_0(z) & b_1(z) & \dots \\ \dots & b_{-3}(z) & b_{-2}(z) & b_{-1}(z) & b_0(z) & \dots \\ & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{7.2}$$

with

$$b_k(z) := \left(\frac{z \pm \sqrt{z^2-4}}{2}\right)^{|k|} \frac{1}{\sqrt{z^2-4}} \text{ for } k \in \mathbb{Z} \text{ and } z \in \rho(\Delta_q) = \mathbb{C}\setminus[-2,2].$$

The sign of $\sqrt{z^2-4}$ should be chosen, such that the inequality $|z \pm \sqrt{z^2-4}| < 2$ is fulfilled.
Moreover, $R_{\Delta_q}(z)$ is a bounded operator and

$$\|R_{\Delta_q}(z)\| \leq \frac{1}{|z^2-4|^{1/2}} \frac{2+|z \pm \sqrt{z^2-4}|}{2-|z \pm \sqrt{z^2-4}|},$$

$z \in \rho(\Delta_q) = \mathbb{C}\setminus[-2,2]$.

Proof: Let $B_q(z)$, with $z \in \mathbb{C}\setminus[-2,2]$, be the right hand side of (7.2), then we have (see e.g. Kato [25], p. 143)

$$\begin{aligned} \|B_q(z)f\|_q &\leq \left(\sum_{k=-\infty}^{\infty} |b_k(z)|\right)^{1-\frac{1}{q}} \left(\sum_{k=-\infty}^{\infty} |b_k(z)|\right)^{\frac{1}{q}} \|f\|_q \\ &= \left(\sum_{k=-\infty}^{\infty} |b_k(z)|\right) \|f\|_q = \frac{1}{|z^2-4|^{\frac{1}{2}}} \left(2\frac{1}{1-\left|\frac{z\pm\sqrt{z^2-4}}{2}\right|} - 1\right)\|f\|_q \\ &= \frac{1}{|z^2-4|^{\frac{1}{2}}} \left(\frac{2+|z \pm \sqrt{z^2-4}|}{2-|z \pm \sqrt{z^2-4}|}\right)\|f\|_q. \end{aligned}$$

We will compute the single matrix-entries of Δ_1 according to the canonical standard basis. For this remember that one can identify $l^1(\mathbb{Z})$ with the space $\mathcal{A}$ which was introduced in Example 6.1. The action of Δ_1 to any function $f \in \mathcal{A}$ is

$$(\Delta_1 f)(z) = \frac{1}{z} f(z) + z f(z).$$

Hence, according to Example 6.1, $\lambda \in \mathbb{C}$ belongs to $\rho(\Delta_q)$ if and only if for every $g \in \mathcal{A}$ there is a unique function $f \in \mathcal{A}$ with

$$((\lambda - \Delta_1)f(z) = g(z) \text{ for all } z \in \partial\mathbb{D},$$

which is equivalent to

$$f(z) = \frac{g(z)}{\lambda - \left(\frac{1}{z} + z\right)} \text{ for all } z \in \partial\mathbb{D}.$$

For every $g \in \mathcal{A}$ the function $z \mapsto \frac{g(z)}{\lambda - \left(\frac{1}{z}+z\right)}$ is an element of $\mathcal{A}$ if and only if

$$\lambda - \left(\frac{1}{z} + z\right) \neq 0 \text{ for all } z \in \partial\mathbb{D} \Leftrightarrow \lambda \notin [-2, 2].$$

Hence, $\sigma(\Delta_1) = \sigma_e(\Delta_1) = [-2, 2]$ (see Proposition 1.6).
The matrix entry at the n-th line and m-th row can be computed with $\langle \delta_n, R_{\Delta_1}(\lambda)\delta_m \rangle$ where δ_j denotes the j-th canonical unit-vector which can be identified with the function $z \mapsto z^j$. Then, due to the Cauchy-Integral formula,

$$\begin{aligned}
\langle \delta_n, R_{\Delta_1}(\lambda)\delta_m \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{e^{i\theta n}} \frac{e^{i\theta m}}{\lambda - (e^{-i\theta} + e^{i\theta})} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta(m-n)}}{\lambda - (e^{-i\theta} + e^{i\theta})} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta|m-n|}}{\lambda - (e^{-i\theta} + e^{i\theta})} d\theta = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{w^{|m-n|}}{\lambda - (w + w^{-1})} \frac{1}{w} dw \\
&= -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{w^{|m-n|}}{\left(w - \frac{\lambda - \sqrt{\lambda^2-4}}{2}\right)\left(w - \frac{\lambda + \sqrt{\lambda^2-4}}{2}\right)} dw \\
&= -\left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}\right)^{|m-n|} \frac{1}{\sqrt{\lambda^2 - 4}},
\end{aligned}$$

where the $\pm$ has to be taken such that

$$\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \in \mathbb{D}$$

which is always possible if $\lambda \notin [-2, 2]$. Hence the assertion follows for Δ_1.
A direct calculation shows for $z \in \mathbb{C} \setminus [-2, 2]$

$$B_q(z)(\Delta_q - z)f = f = (\Delta_q - z)B_q(z)f \text{ for all } f \in l^q(\mathbb{Z}), \quad 1 < q \leq \infty$$

which implies $\rho(L_0) \subseteq \mathbb{C} \setminus [-2,2]$ and therefore $B_q(z) = R_{\Delta_q}(z)$ for all $z \in \mathbb{C} \setminus [-2,2]$.
To prove $\mathbb{C} \setminus [-2,2] = \rho(\Delta_q)$, it suffices to show that for every $\lambda_0 \in [-2,2]$

$$\|R_{\Delta_q}(z)\| \overset{z \to \lambda_0}{\to} \infty.$$

We will show this for the case $q = \infty$:
For every $z \in \mathbb{C} \setminus [-2,2]$ we define $f_z := \left(\frac{|b_k(z)|}{b_k(z)}\right)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$. Then $\|f_z\|_\infty = 1$ for every $z \in \mathbb{C} \setminus [-2,2]$.
Since $|\lambda_0 \pm \sqrt{\lambda_0^2 - 4}| = 2$ for all $\lambda_0 \in [-2,2]$, we have

$$\|R_{\Delta_q}(z)\| \geq \|R_{\Delta_q}(z) f_z\|_\infty \geq \sum_{k=-\infty}^{\infty} |b_k(z)| =$$
$$\frac{1}{|z^2-4|^{\frac{1}{2}}} \left(\frac{2 + |z \pm \sqrt{z^2-4}|}{2 - |z \pm \sqrt{z^2-4}|} \right) \overset{z \to \lambda_0}{\to} \infty.$$

For $q \neq \infty$ one can show this in a similar way if one takes $f = \delta_n$, the canonical nth unit vector. $\square$

7.1.1 Lieb-Thirring inequalities

Let $K \in \Pi_p^{\mathcal{F}}(l^q(\mathbb{Z}))$ and denote

$$L = \Delta_q + K.$$

Since K is compact we have due to Theorem 1.2

$$\sigma_e(L) = \sigma_e(\Delta_q) = [-2,2].$$

According to Theorem 3.14 a holomorphic function on $\hat{\mathbb{C}} \setminus [-2,2]$ is given by

$$d(z) = \det_{\lceil p \rceil}\big(\mathbb{1} - K R_{\Delta_q}(z)\big) \text{ for } z \in \mathbb{C} \setminus [-2,2] \text{ and } d(\infty) = 1.$$

In order to use Theorem 4.4 we take the conformal map (see Section 4.3 third example)

$$\phi(w) = w + w^{-1} \text{ if } w \in \mathbb{D} \setminus \{0\} \text{ and } \phi(0) = \infty$$

which maps $\mathbb{D}$ to $\hat{\mathbb{C}} \setminus [-2,2]$.
Denote $h := d \circ \phi$, then $h(0) = 1$ and (see Theorem 3.4)

$$\log |h(w)| \leq \Gamma_r \|K\|_{\Pi_p}^r \|R_{\Delta_q}(\phi(w))\|^r \text{ for all} w \in \mathbb{D},$$

with $r := \max(2,p)$. For the norm of the resolvent we obtain

$$\begin{aligned}
\|R_{\Delta_q}(\phi(w))\| &\leq \frac{1}{|(w+w^{-1})^2-4|^{\frac{1}{2}}}\left(\frac{2+|w+w^{-1}\pm\sqrt{(w+w^{-1})^2-4}|}{2-|w+w^{-1}\pm\sqrt{(w+w^{-1})^2-4}|}\right) \\
&= \frac{1}{|w-w^{-1}|}\left(\frac{2+|w+w^{-1}\pm\sqrt{(w-w^{-1})^2}|}{2-|w+w^{-1}\pm\sqrt{(w-w^{-1})^2}|}\right) \\
&= \frac{|w|}{|w^2-1|}\left(\frac{2+2|w|}{2-2|w|}\right) \\
&\leq \frac{2|w|}{|w-1||w+1|(1-|w|)}, \quad w \in \mathbb{D}.
\end{aligned}$$

Hence

$$\log|h(w)| \leq 2^r\Gamma_r\|K\|^r_{\Pi_p}\frac{|w|^r}{(1-|w|)^r|w-1|^r|w+1|^r} \text{ for all } w \in \mathbb{D}. \qquad (7.3)$$

Using Theorem 4.4 with $\epsilon = 1-\tau$

$$\sum_{w\in\mathcal{Z}(h)}\frac{(1-|w|)^{r+1+\tau}}{|w|^{r-1+\tau}}|w^2-1|^{r-1+\tau} \leq C(\tau)\|K\|^r_{\Pi_p}$$

with $0<\tau<1$.

For transforming these estimate to an estimate for $\sigma_d(L)$ we use the relations from Theorem 4.5: If $z=\phi(w)$, $w \in \mathbb{D}$. Then we have

$$\frac{1}{2}\frac{|w^2-1|(1-|w|)}{|w|} \leq \mathrm{dist}(z,[-2,2]) \leq \frac{1+\sqrt{2}}{2}\frac{|w^2-1|(1-|w|)}{|w|} \qquad (7.4)$$

and

$$\left|\frac{w^2-1}{w}\right|^2 = |z^2-4|. \qquad (7.5)$$

Theorem 7.2 *Let $L = \Delta_q + K$ be in $l^q(\mathbb{Z})$ with $K \in \Pi_p^{\mathcal{F}}(l^q(\mathbb{Z}))$, $1 \leq p \leq \infty$. Then we get for $\tau > 0$*

$$\sum_{z\in\sigma_d(L)}\frac{\mathrm{dist}(z,[-2,2])^{r+1+\tau}}{|z^2-4|} \leq C(\tau)\|K\|^r_{\Pi_p}, \qquad (7.6)$$

with $r := \max(2,p)$.

Proof: Let $w \in \mathbb{D} \setminus \{0\}$, $z = w + w^{-1}$. By (7.4) and (7.5) we obtain

$$\begin{aligned}(1-|w|)^{r+1+\tau}\left|\frac{w^2-1}{w}\right|^{r-1+\tau} &= \left(\frac{(1-|w|)|w^2-1|}{|w|}\right)^{r+1+\tau}\left|\frac{w}{w^2-1}\right|^2 \\ &\geq \left(\frac{2}{1+\sqrt{2}}\right)^{r+1+\tau}\frac{\operatorname{dist}(z,[-2,2])^{r+1+\tau}}{|z^2-4|}.\end{aligned}$$

□

The result of Theorem 7.2 relies on inequality (7.3). One disadvantage of (7.3) is that it is very rough, since

$$\log|h(w)| \leq \Gamma_r \|KR_{\Delta_q}(\lambda)\|^r_{\Pi_p}$$

seems to be a stronger inequality. In fact, if we concentrate on the class of nuclear Jacobi operators (which is a subclass of $\Pi_p^{\mathcal{F}}(l^1(\mathbb{Z}))$) it is possible to derive a better estimate than (7.3) and therefore a stronger Lieb-Thirring type inequality.
We call an operator $J \in \mathcal{L}\big(l^1(\mathbb{Z})\big)$ **Jacobi operator** if it is defined by

$$Jf := \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & d_{-1} & c_{-1} & & & \\ & & a_0 & d_0 & c_0 & & \\ & & & a_1 & d_1 & c_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} f, \text{ for all } f \in l^1(\mathbb{Z}).$$

Then J is nuclear, if and only if $(a_k), (d_k), (c_k) \in l^1(\mathbb{Z})$ (see Example 2.2). In this case we have

$$\|J\|_{\mathcal{N}} = \sum_{k\in\mathbb{Z}} \sup\{|a_k|, |d_k|, |c_k|\} < \infty.$$

Theorem 7.3 *Let $L = \Delta_1 + J$ be in $l^1(\mathbb{Z})$ with $J \in \mathcal{N}(l^1(\mathbb{Z}))$ a Jacobi operator. Then we get for $\tau > 0$*

$$\sum_{z\in\sigma_d(L)} \operatorname{dist}(z,[-2,2])^{1+\tau} \leq C(\tau)\|J\|^2_{\mathcal{N}}. \tag{7.7}$$

Proof: For every $\lambda \in \rho(\Delta_1)$ the operator $JR_{\Delta_1}(\lambda)$ is given by the matrix

$$-\begin{pmatrix} & \vdots & \vdots & \\ \dots & a_{-1}b_{-1}(\lambda)+d_{-1}b_{-2}(\lambda)+c_{-1}b_{-3}(\lambda) & a_{-1}b_0(\lambda)+d_{-1}b_{-1}(\lambda)+c_{-1}b_{-2}(\lambda) & \dots \\ \dots & a_0b_{-2}(\lambda)+d_0b_{-3}(\lambda)+c_0b_{-4}(\lambda) & a_0b_{-1}(\lambda)+d_0b_{-2}(\lambda)+c_0b_{-3}(\lambda) & \dots \\ \dots & a_1b_{-3}(\lambda)+d_1b_{-4}(\lambda)+c_1b_{-5}(\lambda) & a_1b_{-2}(\lambda)+d_1b_{-3}(\lambda)+c_1b_{-4}(\lambda) & \dots \\ & \vdots & \vdots & \end{pmatrix},$$

where $b_k = \frac{1}{\sqrt{\lambda^2-4}} \left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2} \right)^{|k|}$, where the sign of the square-root has to be taken such that $\left| \frac{\lambda \pm \sqrt{\lambda^2-4}}{2} \right| \leq 1$ (Proposition 7.1). According to Example 2.2 it is possible to estimate the nuclear norm of $JR_{\Delta_1}(\lambda)$:

$$\begin{aligned} \|JR_{\Delta_1}(\lambda)\|_{\mathcal{N}} &= \sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} |a_k b_j(\lambda) + d_k b_{j-1}(\lambda) + c_k b_{j-2}(\lambda)| \\ &\leq \sum_{k \in \mathbb{Z}} \max\{|a_k|, |d_k|, |c_k|\} \sup_{j \in \mathbb{Z}} |3 b_j(\lambda)| \\ &\leq \sum_{k \in \mathbb{Z}} \max\{|a_k|, |d_k|, |c_k|\} 3 \frac{1}{|\sqrt{\lambda^2-4}|} \\ &= \frac{3\|J\|_{\mathcal{N}}}{|\sqrt{\lambda^2-4}|} \end{aligned} \tag{7.8}$$

This implies that (see Theorem 3.4)

$$\log |\underbrace{\det_2(\mathbb{1} - KR_{\Delta_1}(\lambda))}_{:=d(\lambda)}| \leq \frac{1}{2} \|JR_{\Delta_1}(\lambda)\|_{\mathcal{N}} \leq \frac{\frac{9}{2}\|J\|_{\mathcal{N}}^2}{|\lambda^2-4|}.$$

Thus, the holomorphic function $h := d \circ \phi$ satisfies

$$\log |h(w)| \leq \frac{\frac{9}{2}\|J\|_{\mathcal{N}}^2 |w|^2}{|w-1|^2|w+1|^2}, \text{ for all } w \in \mathbb{D}. \tag{7.9}$$

Using Theorem 4.4 with $e = 1 - \tau$ in connection with the inequality $(1 - |w|)^{1+\tau} \leq (1 - |w|)$ we can derive for the zeros of h

$$\sum_{w \in \mathcal{Z}(h)} \frac{(1-|w|)^{1+\tau}}{|w|^{1+\tau}} |w^2 - 1|^{1+\tau} \leq C(\tau) \|J\|_{\mathcal{N}}^2. \tag{7.10}$$

(7.10) together with the right inequality in (7.4) proves the desired assertion. □

Remark 7.4 For results on eigenvalues of Jacobi operators on the Hilbert space $l^2(\mathbb{Z})$ or $l^2(\mathbb{N})$ we refer to [2], [19], [20] or [16].

7.1.2 The closure of the discrete spectrum of Jacobi operators

We start this subsection with an observation on the set of possible accumulation points of the discrete spectrum of the discrete Laplace operator Δ_1 on

$l^1(\mathbb{Z})$ perturbed by a nuclear Jacobi operator J, defined by

$$Jf := \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & d_{-1} & c_{-1} & & & \\ & & a_0 & d_0 & c_0 & & \\ & & & a_1 & d_1 & c_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} f, \text{ for all } f \in l^1(\mathbb{Z}). \qquad (7.11)$$

Corollary 7.5 *Let $L = \Delta_1 + J$. Then $\sigma_d(L)$ does not accumulate to any point of the set*

$$\rho := \{\lambda \in [-2,2] \,:\, 3\|J\|_{\mathcal{N}} < |\lambda^2 - 4|^{\frac{1}{2}}\}.$$

Proof: According to Theorem 5.12 it is sufficient to show that there is an open set $\Omega \subseteq \rho(\Delta_1)$ with

$$\|JR_{\Delta_1}(\lambda)\| < 1 \text{ for all } \lambda \in \Omega$$

and

$$\overline{\Omega} \cap [-2,2] = \rho.$$

In fact, by assumption there has to be an open set $\tilde{\Omega}$ with $\rho \subseteq \tilde{\Omega}$ such that

$$3\|J\|_{\mathcal{N}} < |\lambda^2 - 4|^{\frac{1}{2}} \text{ for all } \lambda \in \tilde{\Omega}.$$

This in connection with inequality (7.8) implies

$$\|JR_{\Delta_1}(\lambda)\| \leq \frac{3\|J\|_{\mathcal{N}}}{|\lambda^2 - 4|^{\frac{1}{2}}}.$$

for all $\lambda \in \Omega := \tilde{\Omega} \setminus \rho$. $\square$

Remark 7.6 Depending on the quantity of $\|J\|_{\mathcal{N}}$ maybe the set ρ can be the empty set.

The next part of this subsection is devoted to a very special class of Jacobi operators. We consider Δ_1 defined on $l^1(\mathbb{Z})$ perturbed by the nuclear Jacobi operator J given by the infinite matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \alpha_{-1} & \beta_{-1} & \alpha_{-1} & & & \\ & & \alpha_0 & \beta_0 & \alpha_0 & & \\ & & & \alpha_1 & \beta_1 & \alpha_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad (7.12)$$

with $\alpha_k = x_k a$, $\beta_k = x_k b$, where $(x_k) \in l^1(\mathbb{C})$ and $a, b \in \mathbb{C}$. For

$$L := \Delta_1 + J$$

it is possible to exclude one special point to be an accumulation point of the discrete spectrum of L.

Corollary 7.7 *Let $L := \Delta_1 + J$. Then the point $-\frac{b}{a}$ is not an accumulation point of $\sigma_{disc}(L)$.*

Proof: According to Theorem 5.10 it suffices to show that $JR_{\Delta_1}(\cdot)$ is continuously extendable to the point $-\frac{a}{b}$ with respect to $\|\cdot\|_{\mathcal{N}}$.
For every $\lambda \in \rho(\Delta_1)$ the operator $JR_{\Delta_1}(\lambda)$ is given by the matrix

$$-\begin{pmatrix} & \vdots & \vdots & \\ \cdots & \alpha_{-1}b_{-1}(\lambda)+\beta_{-1}b_{-2}(\lambda)+\alpha_{-1}b_{-3}(\lambda) & \alpha_{-1}b_0(\lambda)+\beta_{-1}b_{-1}(\lambda)+\alpha_{-1}b_{-2}(\lambda) & \cdots \\ \cdots & \alpha_0 b_{-2}(\lambda)+\beta_0 b_{-3}(\lambda)+\alpha_0 b_{-4}(\lambda) & \alpha_0 b_{-1}(\lambda)+\beta_0 b_{-2}(\lambda)+\alpha_0 b_{-3}(\lambda) & \cdots \\ \cdots & \alpha_1 b_{-3}(\lambda)+\beta_1 b_{-4}(\lambda)+\alpha_1 b_{-5}(\lambda) & \alpha_1 b_{-2}(\lambda)+\beta_1 b_{-3}(\lambda)+\alpha_1 b_{-4}(\lambda) & \cdots \\ & \vdots & \vdots & \end{pmatrix}, \tag{7.13}$$

where $b_k = \frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda\pm\sqrt{\lambda^2-4}}{2}\right)^{|k|}$, where the sign of the square-root has to be taken such that $\left|\frac{\lambda\pm\sqrt{\lambda^2-4}}{2}\right| \leq 1$ (Proposition 7.1).
The matrix (7.13) defines, even if $\lambda \in (-2,2)$, a bounded operator. For brevity this operator will be called J_λ. Since

$$\begin{aligned} \|J_\lambda\|_{\mathcal{N}} &= \sum_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} |\alpha_j b_k(\lambda) + \beta_j b_{k-1}(\lambda) + \alpha_j b_{k-2}(\lambda)| \\ &\leq \sum_{j\in\mathbb{Z}} \sup\{\alpha_j, \beta_j\} \sup_{k\in\mathbb{Z}} (|b_k(\lambda)| + |b_{k-1}(\lambda)| + |b_{k-2}(\lambda)|) \\ &\leq 3\sum_{j\in\mathbb{Z}} \sup\{\alpha_k, \beta_k\} < \infty \text{ for all } \lambda \in (-2,2), \end{aligned}$$

J_λ is a nuclear operator for every $\lambda \in (-2,2)$.
As mentioned in the beginning of this proof it suffices to show that
$JR_{\Delta_1}(\lambda) \overset{\lambda\to-\frac{a}{b}}{\longrightarrow} J_{-\frac{a}{b}}$ with respect to $\|\cdot\|_{\mathcal{N}}$. To do this it is helpful to compute the single entries $e_{kj}(\lambda)$ of the matrix representaion of J_λ for $j-k \neq 0$ (we assume $|j-k-1| > |j-k| > |j-k+1|$, the other case can be treated in

the same way):

$$
\begin{aligned}
e_{k,j}(\lambda) :=& \alpha_k b_{j-k+1}(\lambda) + \beta_k b_{j-k}(\lambda) + \alpha_k b_{j-k-1}(\lambda) = \frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2}\right)^{|j-k-1|} \\
&\times \left(\alpha_k\left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2}\right)^2 + \beta_k \frac{\lambda \pm \sqrt{\lambda^2-4}}{2} + \alpha_k\right) \\
&= \frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2}\right)^{|j-k-1|+1} (\alpha_k \lambda + \beta_k) \\
&= \frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2}\right)^{|j-k-1|+1} x_k(a\lambda + b)
\end{aligned}
$$

For $|j-k| \neq 0$ the value $-\frac{b}{a}$ is a zero of $e_{k,j}$. Using the nuclear norm formula for operators on $l^1(\mathbb{Z})$ we obtain

$$
\begin{aligned}
&\|JR_{\Delta_1}(\lambda) - J_{-\frac{b}{a}}\|_{\mathcal{N}} = \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} |e_{k,j}(\lambda) - e_{k,j}(-\frac{b}{a})| \\
&= \sum_{k\in\mathbb{Z}} \sup \left\{\left|\frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2}\right)^{|j-k-1|+1} x_k(a\lambda+b)\right|\right\}_{j\in\mathbb{Z}\setminus\{k\}, j<k} \\
&\cup \left\{\left|\frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda \pm \sqrt{\lambda^2-4}}{2}\right)^{|j-k+1|+1} x_k(a\lambda+b)\right|\right\}_{j\in\mathbb{Z}\setminus\{k\}, j>k} \\
&\cup \{|e_{k,k}(\lambda) - e_{k,k}\big(-\frac{b}{a}\big)|\} \\
&\leq \sum_{k\in\mathbb{Z}} \sup \left\{\frac{|x_k(a\lambda+b)|}{\sqrt{\lambda^2-4}}\right\} \cup \{|e_{k,k}(\lambda) - e_{k,k}\big(-\frac{b}{a}\big)|\} \\
&\xrightarrow{\lambda \to -\frac{b}{a}} 0.
\end{aligned}
$$

□

7.2 The operator of multiplication

As another easy application we consider $X = C[\alpha, \beta]$ (the space of continuous functions). We define the **operator of multiplication** by

$$M_f : X \to X \text{ with } (M_f g)(t) := f(t)g(t) \text{ for all } g \in C[\alpha, \beta],$$

where f is a real-valued contiuous function on $[\alpha, \beta]$.
We know $M_f \in \mathcal{L}(X), \sigma(M_f) = \sigma_e(M_f) = [\min(f), \max(f)] =: [a, b]$ and

$$\big(R_{M_f}(\lambda)g\big)(x) = \frac{g(x)}{f(x) - \lambda},\ \lambda \in \rho(M_f).$$

In this example it is possible to compute the operator norm of the resolvent explicitly to be

$$\|R_{M_f}(\lambda)\| = \left\|\frac{1}{f - \lambda}\right\|_\infty = \frac{1}{\operatorname{dist}(\lambda, [a, b])}. \tag{7.14}$$

7.2.1 Lieb-Thirring inequalities

Defining an integral operator

$$K : X \to X \text{ with } (Kf)(t) := \int_\alpha^\beta k(t, s) f(s) ds$$

with $k \in C[\alpha, \beta]^2$. According to Example 2.2 this operator is nuclear.
Then for

$$Z := M_f + K$$

the function

$$d(\lambda) := \det\big(\mathbb{1} - K R_{M_f}(\lambda)\big),\ \lambda \in \rho(L_0)$$

is holomorphic with a zero-set equal to $\sigma_d(L)$ and due to (7.14)

$$|d(\lambda)| \leq \frac{1}{2}\|K\|_{\mathcal{N}}^2 \frac{1}{\operatorname{dist}(\lambda, [a, b])^2}.$$

Setting $\phi(w) := \frac{b-a}{4}(w + w^{-1} + 2)$, $w \in \mathbb{D} \setminus \{0\}$, $\phi(0) = \infty$ we receive (for the holomorphic function $d \circ \phi$)

$$|(d \circ \phi)(w)| \leq \frac{1}{2}\|K\|_{\mathcal{N}}^2 \frac{1}{\operatorname{dist}(\phi(w), [a, b])^2}.$$

Once again using the estimate (Theorem 4.5)

$$\frac{b-a}{8}\frac{|w^2 - 1|(1 - |w|)}{|w|} \leq \operatorname{dist}(\phi(w), [a, b]) \leq \frac{(b-a)(1+\sqrt{2})}{8}\frac{|w^2 - 1|(1 - |w|)}{|w|}$$

with $w \in \mathbb{D} \setminus \{0\}$, we obtain

$$|d \circ \phi(w)| \leq \frac{1}{2} C(a,b) \|K\|_{\mathcal{N}}^2 \frac{|w|^2}{|w^2 - 1|^2 (1 - |w|)^2} \tag{7.15}$$

and so by Theorem 4.4

$$\sum_{w \in \mathcal{Z}(d \circ \phi)} \frac{(1 - |w|)^{3+\tau}}{|w|^{1+\tau}} |w^2 - 1|^{1+\tau} \leq C(\tau, a, b) \|K\|_{\mathcal{N}}^2.$$

In analogy to the proof of Theorem 7.2 we can deduce:

Theorem 7.8 *Let $L = M_f + K$ be defined as described above, then*

$$\sum_{\lambda \in \sigma_d(L)} \frac{\operatorname{dist}(\lambda, [a,b])^{3+\tau}}{|\lambda - a||\lambda - b|} \leq C(\tau, a, b) \|K\|_{\mathcal{N}}^2, \ \tau > 0.$$

7.2.2 The closure of the discrete spectrum of a perturbed multiplication operator

Let f be an injective real-valued continuous function.
The set $f(I) \subseteq \sigma_{ess}(M_f)$, where

$$I := \{x \in [\alpha, \beta] | k(t,x) = 0 \text{ for all } t \in [\alpha, \beta]\},$$

plays an important role in the next corollary.

Corollary 7.9 *Let $L := M_f + K$ be defined as in Subsection 7.2.1, then the discrete spectrum of L does not accumulate to any point belonging to $\operatorname{int} f(I)$ (the inner points of $f(I)$ according to $\mathbb{R}$).*

Proof: We assume $\operatorname{int}\big(f(I)\big) \neq \emptyset$ and we take a $\lambda_0 \in \operatorname{int}\big(f(I)\big)$. Obviously, since in this case the function $(t,x) \mapsto k(t,x)(f(x) - \lambda_0)^{-1}$ is a continuous function, there is a nuclear extension of $KR_{M_f}(\cdot)$ from $\rho(M_f)$ to the point λ_0 (let us call this extension K_{λ_0}). According to Theorem 5.10 we have to show that this extension is also continuous. For this let $\lambda \in \rho(M_f)$:

$$\begin{aligned}
&\|KR_{M_f}(\lambda) - K_{\lambda_0}\|_{\mathcal{N}} \\
&\leq \int_\alpha^\beta \sup_{t\in[\alpha,\beta]} |k(t,x)(f(x)-\lambda)^{-1} - k(t,x)(f(x)-\lambda_0)^{-1}|dx \\
&= \int_\alpha^\beta \sup_{t\in[\alpha,\beta]} |k(t,x)\big((f(x)-\lambda)^{-1} - (f(x)-\lambda_0)^{-1}\big)|dx \\
&= \int_\alpha^\beta |(f(x)-\lambda)^{-1} - (f(x)-\lambda_0)^{-1}|\chi_{[\alpha,\beta]\setminus I} \sup_{t\in[\alpha,\beta]} |k(t,x)|dx \\
&\leq \sup_{\xi\in[\alpha,\beta]\setminus I} |(f(\xi)-\lambda)^{-1} - (f(\xi)-\lambda_0)^{-1}| \int_\alpha^\beta \sup_{t\in[\alpha,\beta]} |k(t,x)|dx \overset{\lambda\to\lambda_0}{\longrightarrow} 0.
\end{aligned}$$

The function χ_M defines the characteristic function on the set M, i.e. $\chi_M(x) = 1$ if $x \in M$ and else $\chi_M(x) = 0$.
Hence, the map $KR_{M_f}(\cdot)$ is continuously extendable to $\operatorname{int}\big(f(I)\big)$. □

7.3 Shift-operators

As third application let us consider the shift operator as the free operator. In Example 6.1 the shift operator, $S : l^1(\mathbb{Z}) \to l^1(\mathbb{Z})$ defined by $(Sf)(n) := f(n-1)$ for all $f \in l^1(\mathbb{Z})$ was already discussed. It turned out, that the resolvent set of S is the disjoint union of $\mathbb{D}$ (the bounded component) and
$\mathbb{C} \setminus \overline{\mathbb{D}}$ (the unbounded component). Similar to the resolvent of normal operators, the norm of $R_S(\lambda)$ can be estimated in terms of the distance of λ to the spectrum of S.

Lemma 7.10 *Let S be the shift operator on the line, then*

$$\|R_S(\lambda)\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(S))} = \frac{1}{||\lambda|-1|}, \lambda \in \rho(S).$$

Proof: For $|\lambda| < 1$:

$$\|R_S(\lambda)\| = \|\sum_{k=0}^{\infty} \lambda^k S^{-k-1}\| \leq \sum_{k=0}^{\infty} |\lambda|^k \underbrace{\|S^{-1}\|^{k+1}}_{=1} = \frac{1}{1-|\lambda|}.$$

For $|\lambda| > 1$:

$$\|R_S(\lambda)\| = \left\| \sum_{k=0}^{\infty} \frac{S^k}{\lambda^{k+1}} \right\| \le \sum_{k=0}^{\infty} \frac{1}{|\lambda|^{k+1}} \underbrace{\|S\|^k}_{=1} = \frac{1}{|\lambda| - 1}.$$

$\square$

Moreover, according to the canonical standard basis $(\delta_n)_{n\in\mathbb{Z}}$ it is possible to compute the matrix representation of $R_S(\lambda)$. The entry in the $n-$th line and $m-$th row of this representation is given by

$$\begin{aligned}
\langle \delta_n, R_S(\lambda)\delta_m \rangle &= \begin{cases} \left\langle \delta_n, \sum_{k=0}^{\infty} \frac{S^k}{\lambda^{k+1}} \delta_m \right\rangle & \text{if } |\lambda| > 1 \\ -\left\langle \delta_n, \sum_{k=0}^{\infty} \lambda^k S^{-k-1} \delta_m \right\rangle & \text{if } |\lambda| < 1 \end{cases} \\
&= \begin{cases} \sum_{k=0}^{\infty} \left\langle \delta_n, \frac{\delta_{m+k}}{\lambda^{k+1}} \right\rangle = \begin{cases} \frac{1}{\lambda^{m-n+1}} & \text{if } n \ge m, \\ 0 & \text{if } n < m, \end{cases} & \text{if } |\lambda| > 1 \\ -\sum_{k=0}^{\infty} \langle \delta_n, \lambda^k \delta_{m-k-1} \rangle = \begin{cases} 0 & \text{if } n > m-1, \\ \lambda^{n-m+1} & \text{if } n \le m-1 \end{cases} & \text{if } |\lambda| < 1. \end{cases}
\end{aligned}$$

Hence, $R_S(\lambda)$ is given by the infinite matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ \cdots & \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & & \\ \cdots & \frac{1}{\lambda^4} & \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & \\ \cdots & \frac{1}{\lambda^5} & \frac{1}{\lambda^4} & \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

if $|\lambda| > 1$ (the main diagonal is the diagonal with the entries $\frac{1}{\lambda}$),
and for $|\lambda| < 1$ $R_S(\lambda)$ is given by the infinite matrix

$$-\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & 0 & 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \cdots \\ & & 0 & 1 & \lambda & \lambda^2 & \lambda^3 & \cdots \\ & & & 0 & 1 & \lambda & \lambda^2 & \cdots \\ & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

if $|\lambda| < 1$ (the main diagonal is the diagonal with the entries 0).

7.3.1 Lieb-Thirring inequalities

Let $p > 1$, $K \in \Pi_p^{\mathcal{F}}(l^q(\mathbb{Z}))$ be a p-summing operator and set

$$L := S + K.$$

Hence, according to Theorem 3.15 the set of eigenvalues of L coincides with the zeros of the holomorphic function

$$d(\lambda) := \det_{\lceil p \rceil}\left(\mathbb{1} - KR_S(\lambda)\right), \lambda \in \rho(S)$$

and

$$|d(\lambda)| \leq \exp\left(\frac{\Gamma_r \|K\|_{\Pi_p}^r}{||\lambda| - 1|^r}\right) \text{ for all } \lambda \in \rho(S), \tag{7.16}$$

with $r := \max(2, p)$.
Via the conformal map $\phi : \mathbb{D} \to \mathbb{D}^c$, defined by $\phi(z) := \frac{1}{z}$, inequality (7.16) becomes

$$|(d \circ \phi)(z)| \leq \exp\left(\frac{\Gamma_r \|K\|_{\Pi_p}^r |z|^r}{(1 - |z|)^r}\right) \text{ for all } z \in \mathbb{D}. \tag{7.17}$$

A consequence of this inequality is:

Theorem 7.11 *Let $L = S + K$ be defined on $l^q(\mathbb{Z})$ with $K \in \Pi_p^{\mathcal{F}}(l^q(\mathbb{Z}))$. Then we get for $1 > \tau > 0$*

$$\sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}^c} \operatorname{dist}(\lambda, \sigma(S))^{r+\tau+1} < C(K, \tau, p)\|K\|_{\Pi_p}^r, \tag{7.18}$$

with $r := \max(2, p)$ and C is a constant only depending on K, τ and p.

Proof: Due to Theorem 4.4 ($\epsilon := 1 - \tau$) and (7.17):

$$\sum_{z \in \mathcal{Z}(d \circ \phi)} \frac{(1 - |z|)^{r+1+\tau}}{|z|^{r+\tau}} \leq C(\tau, p)\|K\|_{\Pi_p}^r. \tag{7.19}$$

Since $z \in \mathcal{Z}(d \circ \phi)$ if and only if $z = \frac{1}{\lambda}$ with $\lambda \in \sigma_d(L) \cap \mathbb{D}^c$, and the modulus of each eigenvalue is bounded by $\|K\| + \|S\| = \|K\| + 1$, it follows that

$$\sum_{z \in \mathcal{Z}(d \circ \phi)} \frac{(1 - |z|)^{r+1+\tau}}{|z|^{r+\tau-1}} = \sum_{\lambda \in \sigma(L) \cap \mathbb{D}^c} \frac{(|\lambda| - 1)^{r+1+\tau}}{|\lambda|^2} \geq \sum_{\lambda \in \sigma(L) \cap \mathbb{D}^c} \frac{(|\lambda| - 1)^{r+1+\tau}}{(\|K\| + 1)^2}. \tag{7.20}$$

Inequality (7.19) and inequality (7.20) together confirm (7.18). □

A similar result as stated in Theorem 7.11 is also possible for the eigenvalues of L in the bounded component of $\rho(S)$, if we additionaly assume that 0 is no eigenvalue of L.

Theorem 7.12 *Let $L = S + K$ be defined on $l^q(\mathbb{Z})$ with $K \in \Pi_p^{\mathcal{F}}(l^q(\mathbb{Z}))$ and $0 \notin \sigma(L)$. Then for $1 > \tau > 0$*

$$\sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}} \mathrm{dist}(\lambda, \sigma(S))^{r+\tau+1} < C \|K\|_{\Pi_p}^r, \tag{7.21}$$

with $r := \max(2, p)$ and C is a constant only depending on K, τ and p.

Proof: $\mathbb{D} \ni \lambda \mapsto d(\lambda) := \det_{\lceil p \rceil}(\mathbb{1} - KR_S(\lambda))$ is a holomorphic function on the open unit disc, the zeros of which coincide with the discrete spectrum of L in $\mathbb{D}$, since $d(0) \neq 0$ the spectrum of L in $\mathbb{D}$ is purely discrete (see Theorem 6.2). Without loss of generality one can assume, that $d(0) = 1$.[1] Moreover, due to Lemma 7.10

$$|d(\lambda)| \leq \exp\left(\frac{\Gamma_r \|K\|_{\Pi_p}^r}{(1 - |\lambda|)^r}\right) \text{ for } \lambda \in \mathbb{D},$$

and by Theorem 4.4

$$\sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}} \frac{\mathrm{dist}\,(\lambda, \sigma(S))^{r+\tau+1}}{|\lambda|^{r+\tau-1}} = \sum_{\lambda \in \mathcal{Z}(d)} \frac{(1 - |\lambda|)^{r+\tau+1}}{|\lambda|^{r+\tau-1}} \leq C(\tau, p) \|K\|_{\Pi_p}^r. \tag{7.22}$$

Since $0 \notin \sigma(L)$,

$$C(K) := \inf\{|\lambda|^{r+\tau+1} : \lambda \in \sigma_d(L)\} > 0$$

and therefore

$$\sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}} \frac{\mathrm{dist}\,(\lambda, \sigma(S))^{r+\tau+1}}{|\lambda|^{r+\tau-1}} \geq \sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}} \frac{\mathrm{dist}\,(\lambda, \sigma(S))^{r+\tau+1}}{C(K)}. \tag{7.23}$$

[1] if $d(0) \neq 0$ then $\tilde{d} := \frac{d}{d(0)}$ is a holomorphic function the zeros of which coincide with the discrete eigenvalues of L in $\mathbb{D}$ and

$$|\tilde{d}(\lambda)| \leq \exp\left(\log\left|\frac{1}{d(0)}\right| + \frac{\|K\|_{\Pi_p}^p}{(1 - |\lambda|)^p}\right) \leq \exp\left(\frac{\log\left|\frac{1}{d(0)}\right| + \|K\|_{\Pi_p}^p}{(1 - |\lambda|)^p}\right)$$
$$\leq \exp\left(C\frac{\|K\|_{\Pi_p}^p}{(1 - |\lambda|)^p}\right) \text{ for all } \lambda \in \mathbb{D}$$

with $C := \frac{1}{\|K\|_{\Pi_p}^p} \log\left|\frac{1}{d(0)}\right| + 1$.

Inequality (7.22) and inequality (7.23) verify (7.21). $\square$

Note that one obtains a similar result as in Theorem 7.11 if the perturbation operator K is an operator of type l^p, i.e. K has p-summing approximation numbers, which follows by Corollary 5.6. In this case ($K \in \mathcal{S}_p(l^q(\mathbb{Z}))$, $p > 0$) one has

$$\sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}^c} (|\lambda| - \|S\|)^q = \sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}^c} \operatorname{dist}(\lambda, \sigma(S))^q < \infty$$

for each $q > p + 1$.

The situation will change if we concentrate on a more explicit kind of p-summing operators. Assume K to be the nuclear diagonal operator on $l^1(\mathbb{Z})$, defined by the infinite matrix

$$\begin{pmatrix} \ddots & & & & \\ & a_{-1} & & & \\ & & a_0 & & \\ & & & a_1 & \\ & & & & \ddots \end{pmatrix}. \tag{7.24}$$

according to the canonical standard basis, with $(a_n)_{n \in \mathbb{Z}} \in l^1(\mathbb{Z})$.

Theorem 7.13 *Let $L = S + K$ defined on $l^1(\mathbb{Z})$ and K the nuclear operator defined by (7.24). Then*

$$\sum_{\lambda \in \sigma_d(L) \cap \mathbb{D}^c} \operatorname{dist}(\lambda, \sigma(S)) \leq \frac{1}{2} \|(a_n)\|_{l^1}^2.$$

Proof: The operator $KR_S(\lambda)$ is given by

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ \cdots & \frac{a_{-1}}{\lambda^3} & \frac{a_{-1}}{\lambda^2} & \frac{a_{-1}}{\lambda} & & \\ \cdots & \frac{a_0}{\lambda^4} & \frac{a_0}{\lambda^3} & \frac{a_0}{\lambda^2} & \frac{a_0}{\lambda} & \\ \cdots & \frac{a_1}{\lambda^5} & \frac{a_1}{\lambda^4} & \frac{a_1}{\lambda^3} & \frac{a_1}{\lambda^2} & \frac{a_1}{\lambda} \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

for $|\lambda| > 1$.

The nuclear norm of $KR_S(\lambda)$ can be estimated by

$$\|KR_S(\lambda)\|_{\mathcal{N}} = \sum_{k=-\infty}^{\infty} \sup_{j \in \mathbb{N}} \left| \frac{a_k}{\lambda^j} \right| \leq \sum_{k=-\infty}^{\infty} |a_k| = \|(a_n)\|_{l^1} < \infty.$$

The zeros of the holomorphic function $\overline{\mathbb{D}}^c \ni \lambda \mapsto d(\lambda) := \det_2(\mathbb{1} - KR_S(\lambda)$ coincide with $\sigma_d(L) \cap \overline{\mathbb{D}}^c$ and d can be estimated due to

$$\begin{aligned}|d(\lambda)| = |\det_2(\mathbb{1} - KR_S(\lambda))| &\leq \exp\left(\frac{1}{2}\|KR_S(\lambda)\|^2_{\mathcal{N}}\right) \\ &\leq \exp\left(\frac{1}{2}\|(a_n)\|^2_{l^1}\right) \text{ for all } \lambda \in \overline{\mathbb{D}}^c.\end{aligned}$$

Thus d is bounded on $\mathbb{D}^c$. Consequently, due to Corollary 4.2, for the zeros of $d \circ \phi$ with $\phi(z) = \frac{1}{z}$ for all $z \in \mathbb{D}$ it follows

$$\sum_{\lambda \in \sigma_d(L) \cap \overline{\mathbb{D}}^c} \frac{|\lambda| - 1}{|\lambda|} = \sum_{z \in \mathcal{Z}(d \circ \phi)} (1 - |z|) \leq \frac{1}{2}\|(a_n)\|^2_{l^1},$$

where the left handside can be estimated by (see also 7.20)

$$\sum_{\lambda \in \sigma_d(L) \cap \overline{\mathbb{D}}^c} \frac{|\lambda| - 1}{|\lambda|} \geq \sum_{\lambda \in \sigma_d(L) \cap \overline{\mathbb{D}}^c} \frac{|\lambda| - 1}{\|K\| + 1}.$$

□

7.3.2 The closure of the discrete spectrum for perturbations of the shift operator on on $l^1(\mathbb{N})$

The shift operator on $l^1(\mathbb{N})$ is defined by $(Sf)(n) := f(n-1)$ if $n > 1$ and $(Sf)(1) = 0$. Its matrix representation according to (δ_n) is given by

$$\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

The space $l^1(\mathbb{N})$ can be identified with the space $\tilde{\mathcal{A}} := \{\sum_{n=0}^{\infty} a_n z^n : (a_n) \in l^1(\mathbb{N}), z \in \mathbb{D}\}$ via the embedding $\psi : l^1(\mathbb{N}) \to \tilde{\mathcal{A}}$ defined by $\psi((a_n)) := z \mapsto \sum_{n=0}^{\infty} a_n z^n$. Then, similar to Example 6.1, the effect of S to any function $f \in \tilde{\mathcal{A}}$ is

$$(Sf)(z) = zf(z) \text{ for all } z \in \mathbb{D}.$$

Thus $\lambda \in \mathbb{C}$ belongs to $\rho(S)$ if and only if for every $g \in \tilde{\mathcal{A}}$ there is a unique $f \in \tilde{\mathcal{A}}$ with

$$((\lambda \mathbb{1} - S)f)(z) = g(z) \text{ for all } z \in \mathbb{D},$$

which is equivalent to

$$f(z) = \frac{g(z)}{\lambda - z}.$$

For every $g \in \tilde{\mathcal{A}}$ the function $z \mapsto \frac{g(z)}{\lambda - z}$ is an element of $\tilde{\mathcal{A}}$ if and only if $\lambda \notin \overline{\mathbb{D}}$ and therefore $\rho(S) = \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\sigma(S) = \overline{\mathbb{D}}$.
Moreover, a direct computation shows, that the infinite matrix

$$\begin{pmatrix} \frac{1}{\lambda} & & & \\ \frac{1}{\lambda^2} & \frac{1}{\lambda} & & \\ \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

for $|\lambda| > 1$, defines the resolvent $R_S(\lambda)$ according to the canonical standard basis.
For any $0 < a < 1$ we define K to be the nuclear operator defined by the infinite matrix

$$\begin{pmatrix} 1 & & & & \\ & a^1 & & & \\ & & a^2 & & \\ & & & a^3 & \\ & & & & \ddots \end{pmatrix}. \tag{7.25}$$

according to the canonical standard basis.
One can make an assertion on the set of accumulation points of the discrete eigenvalues of $L = S + K$.

Theorem 7.14 *Let $L = S + K$ be an operator on $l^1(\mathbb{N})$ and K defined by the matrix (7.25). Then $\partial(\sigma_d(L) \cap \mathbb{D}^c) \cap \partial\mathbb{D} = \emptyset$.*

Proof: According to the canonical standard basis of $l^1(\mathbb{N})$ the matrix

$$\begin{pmatrix} \frac{1}{\lambda} & & & & \\ \frac{a}{\lambda^2} & \frac{a}{\lambda} & & & \\ \frac{a^2}{\lambda^3} & \frac{a^2}{\lambda^2} & \frac{a^2}{\lambda} & & \\ \frac{a^3}{\lambda^4} & \frac{a^3}{\lambda^3} & \frac{a^3}{\lambda^2} & \frac{a^3}{\lambda} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.26}$$

defines an operator K_λ (not necessaraly bounded) for every $\lambda \neq 0$.
If $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ then

$$K_\lambda = KR_S(\lambda) \tag{7.27}$$

and therefore K_λ is a nuclear operator. If $0 < |\lambda| \leq 1$ then

$$\|K_\lambda\|_{\mathcal{N}} = \sum_{k=0}^{\infty} \sup_{1 \leq j \leq k+1} \left| \frac{a^k}{\lambda^j} \right| = \sum_{k=0}^{\infty} \frac{|a|^k}{|\lambda|^{k+1}} = \frac{1}{|\lambda|} \sum_{k=0}^{\infty} \left| \frac{a}{\lambda} \right|^k . \tag{7.28}$$

(7.28) is finite for all λ with $a < |\lambda| \leq 1$. Thus

$$\mathbb{C} \setminus \overline{\mathbb{D}} \ni \lambda \mapsto K_\lambda$$

is analytic and nuclear valued. Hence

$$\lambda \mapsto \det{}_2(\mathbb{1} - K_\lambda)$$

is holomorphic on $\mathbb{C} \setminus (a\overline{\mathbb{D}})$, which implies that the zeros of this function do not accumulate at $\partial\mathbb{D}$.
But (7.27) also implies that

$$\det{}_2(\mathbb{1} - K_\lambda) = \det{}_2\left(\mathbb{1} - KR_S(\lambda)\right) \text{ for all } \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Since the discrete eigenvalues of L in $\mathbb{C} \setminus \overline{\mathbb{D}}$ coinicide with the zeros of $\lambda \mapsto \det_2\left(\mathbb{1} - KR_S(\lambda)\right))$, they cannot accumulate at $\partial\mathbb{D}$.

$\square$

Chapter 8

Determinants of infinite order for operators on finite dimensional spaces

This chapter deals with determinants of infinite order. At the beginning we introduce this topic for general Banach spaces and present with an example the problems with this kind of determinants. The goal of this chapter is to give a solution for finite dimensional spaces.
As in Chapter 3.5 we assume that K is a compact operator on a Banach space X with the property

$$\alpha_n(K) \overset{n\to\infty}{\to} 0. \tag{8.1}$$

In Chapter 3.5 the eigenvalues of

$$L = L_0 + K \quad (L_0 \text{ bounded})$$

in a domain $\Omega \subseteq \hat{\rho}(L_0)$ with $\infty \in \Omega$ and $\overline{\Omega} \cap \sigma(L_0) = \emptyset$ were identified with the zeros of a holomorphic function. This was realized involving a finite dimensional reduction argument and pth regularized determinants.
If we assume, in addition to (8.1), that $K \in \mathcal{S}_{(p_n)}(X)$ where $(p_n) \subseteq \mathbb{N}$ is monotone, and denote for each $z \in \rho(L_0)$ by $(\lambda_n(z))$ a sequence of arbitrary order of the eigenvalues of $KR_{L_0}(z)$, then

$$d(z) := \det{}_{(p_n),(\lambda_n(z))}\left(\mathbb{1} - KR_{L_0}(z)\right) = \prod_{i=1}^{\infty}(1-\lambda_i(z))\exp\left(\sum_{j=1}^{p_i-1}\frac{\lambda_i(z)^j}{j}\right), z \in \rho(L_0)$$

defines a function the zeros of which coincide with the discrete eigenvalues of L.
In contrast to the previous cases, the assertion below is in general not true:

If $z \mapsto K(z) \in \mathcal{S}_{(p_n)}(X)$ is analytic on Ω with eigenvalue sequence $(\lambda_n(z))$, then $z \mapsto \det_{(p_n),(\lambda_n(z))}(\mathbb{1} - K(z))$ is holomorphic on Ω.

Example 8.1 Let $X = \mathbb{C}^3$ and for every $z \in \mathbb{C}$ let $K(z)$ be the operator defined by the matrix

$$\begin{pmatrix} 0 & z & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z+1 \end{pmatrix}.$$

The map $z \mapsto K(z)$ is analytic on the whole complex plane. For each $z \in \mathbb{C}$ a sequence of eigenvalues of $K(z)$ is given by $(\lambda_1(z), \lambda_2(z), \lambda_3(z))$, where $\lambda_1(z) = \sqrt{z}$ and $\lambda_2(z) = -\sqrt{z}$ and $\lambda_3(z) = z + 1$. If we define $p_n = n$ for $n \in \mathbb{N}$ then

$$\det_{(p_n),(\lambda_n(z))}(\mathbb{1} - K(z)) = z(z-1)\exp(-\sqrt{z} + \frac{z+1}{2}) \text{ for all } z \in \mathbb{C}.$$

But this function fails to be holomorphic.

This section deals with this subject in finite dimensional spaces.

Let X be an n-dimensional space, $\Omega \subseteq \mathbb{C}$ open and connected and $\Omega \ni z \mapsto K(z)$ an analytic family of operators acting on X. It follows (see e.g. [25] p. 64) that the number of distinct eigenvalues of $K(z)$ is constant $d \leq n$ independent on z, with the exception of a discrete number of **exceptional points** in Ω. I.e. if z_0 is an exceptional point, then the number of eigenvalues of $K(z_0)$ is less than d.

Theorem 8.2 ([25] p. 65) *Let X, Ω and $\Omega \ni z \mapsto K(z)$ be as in the beginning of this section. Assume that Ω_0 is a simply connected subdomain of Ω containing no exceptional points, then the eigenvalues of $K(z)$ can be written by*

$$\lambda_1(z), \ldots, \lambda_d(z), d \leq n, z \in \Omega_0,$$

where $z \mapsto \lambda_i(z)$ is holomorphic on Ω_0 for each $1 \leq i \leq d$.

As a consequence we have:

Corollary 8.3 *Let X, Ω, Ω_0 and $\lambda_1(z), \ldots, \lambda_d(z)$ be as in Theorem 8.2. Let $p_1, \ldots, p_d \in \mathbb{N}$ with $p_i \leq p_{i+1}$, then*

$$d(z) := \prod_{i=1}^{d} (1 - \lambda_i(z)) \exp\left(\sum_{j=1}^{p_i - 1} \frac{\lambda_i(z)^j}{j}\right), \quad z \in \Omega_0 \tag{8.2}$$

defines a holomorphic function on Ω_0.

If $z_0 \in \Omega$ is an exceptional point then in general it is not possible to define holomorphic functions $z \mapsto \lambda_1(z), \ldots, z \mapsto \lambda_d(z)$ on a neighbourhood of z_0 which coincide with the eigenvalues of $K(z)$, and therefore a construction as in (8.2) fails to be holomorphic (see Example 8.1).
It seems that the exceptional points play an important role in this topic. Therefore we will explain this term in a more illustrative way.
Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\Omega \ni z \mapsto K(z)$ be an analytic family of operators on X. Let $\lambda_1(\cdot), \ldots, \lambda_d(\cdot)$, $d \leq n$, complex (not necessarily continuous) functions on Ω with $\{\lambda_1(z), \ldots, \lambda_d(z)\} = \sigma(K(z))$. Then, $z_0 \in \Omega$ is an exceptional point if and only if there exists two distinct indices $i_0, j_0 \in \{1, \ldots, d\}$ such that $\lambda_{i_0}(z_0) = \lambda_{j_0}(z_0)$. The values $\mu_1, \ldots, \mu_l$, $l < d$, denote the distinct eigenvalues of $K(z_0)$, i.e. $\{\mu_1, \ldots, \mu_l\} = \{\lambda_1(z_0), \ldots, \lambda_d(z_0)\}$. Therefore, the set of indices may be grouped in the following manner

$$P_{\lambda(z_0)} := \{s_1, \ldots, s_l\}$$

with

$$s_i := \{j \in \{1, \ldots, d\} \,:\, \lambda_j(z_0) = \mu_i\} \text{ for all } i \in \{1, \ldots, l\}.$$

Hence, $P_{\lambda(z_0)}$ is a partition of the index set $\{1, \ldots, d\}$. By

$$\mathbb{P}_{\lambda(z_0)} := \{\pi : \{1, \ldots, d\} \to \{1, \ldots, d\} \,:\, \pi(s_i) = s_i, i = 1, \ldots, l\}$$

we denote the set of all permutations which are invariant under all elements of $P_{\lambda(z_0)}$.
A function $f : \mathbb{C}^d \to \mathbb{C}$ is called **symmetric with respect to** $P_{\lambda(z_0)}$ if and only if

$$f(z_1, \ldots, z_d) = f(z_{\pi(1)}, \ldots, z_{\pi(d)}) \text{ for all } \pi \in \mathbb{P}_{\lambda(z_0)}.$$

Theorem 8.4 ([11] Theorem 2.1) *Let $\Omega \subseteq \mathbb{C}$ and $\Omega \ni z \mapsto K(z)$ be as in the beginning of this section, let $z_0 \in \Omega$ be an exceptional point and let $\{\lambda_1(z), \ldots, \lambda_d(z)\} = \sigma(K(z))$ for every $z \in \Omega$ such that the functions $\lambda_1(\cdot), \ldots, \lambda_d(\cdot)$ are continuous at z_0. If $f : \mathbb{C}^d \to \mathbb{C}$ is an analytic function which is symmetric with respect to $P_{\lambda(z_0)}$, then there is a neighbourhood $\mathcal{U}(z_0)$ of z_0 such that*

$$\Omega \ni z \mapsto f(\lambda_1(z), \ldots, \lambda_d(z))$$

is holomorphic on $\mathcal{U}(z_0)$.

A direct consequence is the corollary below.

Corollary 8.5 *Let $\Omega \subseteq \mathbb{C}$ and $\Omega \ni z \mapsto K(z)$ be as in the beginning of this section, $z_0 \in \Omega$ be an exceptional point and let $\{\lambda_1(z), \ldots, \lambda_d(z)\} = \sigma(K(z))$ for every $z \in \Omega$ such that the functions $\lambda_1(\cdot), \ldots, \lambda_d(\cdot)$ are continuous at z_0. Denote $P_{\lambda(z_0)} := \{s_1, \ldots, s_l\}$ with $l < d$. If $p_1, \ldots, p_l$ satisfy $p_1 \leq \ldots \leq p_l$ then there exists a neighbourhood $\mathcal{U}(z_0)$ of z_0 such that*

$$\Omega \ni z \mapsto \prod_{i=1}^{l} \prod_{k \in s_i} (1 - \lambda_k(z)) \exp\left(\sum_{j=1}^{p_i - 1} \frac{\lambda_k(z)^j}{j} \right)$$

is holomorphic on $\mathcal{U}(z_0)$.

Proof: Define

$$f(\lambda_1, \ldots, \lambda_d) := \prod_{i=1}^{l} \prod_{k \in s_i} (1 - \lambda_k) \exp\left(\sum_{j=1}^{p_i - 1} \frac{\lambda_k^j}{j} \right) \text{ for } \lambda_1, \ldots, \lambda_d \in \mathbb{C}.$$

f is analytic and symmetric with respect to the partition $P_{\lambda(z_0)}$. According to Theorem 8.4 there is a neighbourhood $\mathcal{U}(z_0)$ of z_0 such that

$$z \mapsto f\left(\lambda_1(z), \ldots, \lambda_d(z)\right)$$

is holomorphic on $\mathcal{U}(z_0)$. □

Remark 8.6 According to Example 8.1 an exceptional point of the analytic operator valued function $z \mapsto K(z)$ defined by the matrix

$$\begin{pmatrix} 0 & z & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z+1 \end{pmatrix}$$

is the value 0. In fact $\lambda_1(0) = 0 = \lambda_2(0) \neq \lambda_3(0)$, and therefore a suitable partition is $P_{\lambda(0)} := \{s_1, s_2\}$ with $s_1 := \{1, 2\}$ and $s_3 := \{3\}$.
If one defines $p_1, p_2 = 1$ and $p_3 = 2$ one has

$$d(z) = \det_{(p_n),(\lambda_n(z))} (\mathbb{1} - K(z)) = z(z-1) \exp(z+1) \text{ for } z \in \mathbb{C}.$$

Then d is holomorphic in a neighbourhood of 0 (d is even an entire function) which confirms the assertion of Corollary 8.5.

Corollary 8.7 *Let L_0 and K be operators acting on the n dimensional space X. Denote*

$$L = L_0 + K$$

Let $z_0 \in \rho(L_0)$ be an exceptional point of the operator valued map $\rho(L_0) \ni z \mapsto KR_{L_0}(z)$. For every $z \in \rho(L_0)$ let $\{\lambda_1(z), \ldots, \lambda_n(z)\} = \sigma\left(KR_{L_0}(z)\right)$ be eigenvalues counted with their algebraic multiplicity and continuous at z_0. Denote $P_{\lambda(z_0)} := \{s_1, \ldots, s_d\}$. Assume that $j_1 \in s_{i_1}, j_2 \in s_{i_2}$ imply $j_1 < j_2$ whenever $i_1 < i_2$. Let $q_1 \leq \ldots \leq q_d$ be positive integers and define $p_j := q_i$ if $j \in s_i$, then there exists a neighbourhood $\mathcal{U}(z_0)$ of z_0 such that

$$d(z) := \det_{(p_n),(\lambda_n(z))}\left(\mathbb{1} - KR_{L_0}(z)\right) \text{ for } z \in \rho(L_0)$$

defines a holomorphic function on $\mathcal{U}(z_0)$.
Moreover, $z \in \sigma(L) \cap (\mathcal{U}(z_0))$ with algebraic multiplicity $m_L(z) = m$ if and only if $z \in \mathcal{Z}(d|_{\mathcal{U}(z_0)})$ with order $o_d(z) = m$.

Proof: The holomorphicity is a direct consequence of Corollary 8.7.
To prove the assertion concerning the multiplicity of the eigenvalues in $\mathcal{U}(z_0)$ note that

$$d(z) = \underbrace{\det_1(\mathbb{1} - KR_{L_0}(z))}_{=:g(z)} \underbrace{\exp\left(\sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\frac{\lambda_i(z)^j}{j}\right)}_{=:h(z)} \text{ for all } z \in \rho(L_0).$$

Then g is holomorphic on $\rho(L_0)$ (see e.g. Theorem 3.8). The function

$$(\lambda_1, \ldots, \lambda_n) \mapsto \exp\left(\sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\frac{\lambda_i^j}{j}\right)$$

is an analytic function and symmetric with respect to $P_{\lambda(z_0)}$. Thus

$$z \mapsto h(z) = \exp\left(\sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\frac{\lambda_i(z)^j}{j}\right)$$

is holomorphic in a neighbourhood of z_0.
Moreover, $h(z) \neq 0$ for all $z \in \rho(L_0)$. It follows (see Corollary 3.10)

$$\begin{aligned} & z \in \sigma_d(L) \cap \mathcal{U}(z_0) \text{ with } m_L(z) = m \\ \Leftrightarrow & z \in \mathcal{Z}(g|_{\mathcal{U}(z_0)}) \text{ with } o_g(z) = m \\ \Leftrightarrow & z \in \mathcal{Z}(d|_{\mathcal{U}(z_0)}) \text{ with } o_d(z) = m \end{aligned}$$

□

Chapter 9

Open problems and additional remarks

In this final section some open problems in the context of the results presented in this thesis will be discussed.

(1) One possible next step is to extend the subject of this thesis, i.e. the study of the discrete spectrum of bounded operators to densely defined closed operators. For the Hilbert space case this is already done e.g. in [6]. If we assume that L and L_0 are closed densely defined operators in a Banach space X such that

   (a) $\rho(L_0) \cap \rho(L) = \emptyset$,

   (b) $R_L(b) - R_{L_0}(b)$ is compact for some $b \in \rho(L) \cap \rho(L_0)$,

   (c) $\rho(L_0) \cap \sigma(L) = \sigma_d(L)$.

   then there is the relation

$$\lambda \in \sigma_d(L) \Leftrightarrow 1 \in \sigma_d \left((R_L(a) - R_{L_0}(a)) \left(\lambda - R_{L_0}(a) \right)^{-1} \right.$$

   where $a \in \rho(L) \cap \rho(L_0)$.
   Moreover, if $R_L(b) - R_{L_0}(b)$ belongs to one of the quasi-Banach ideals discussed in this thesis, we can identify the eigenvalues of L with the zeros of

$$\lambda \mapsto \det_{\lceil p \rceil} \left(\mathbb{1} - \left((R_L(a) - R_{L_0}(a)) \left(\lambda - R_{L_0}(a) \right)^{-1} \right) \right).$$

(2) One problem arises from Corollary 5.3 where the number of eigenvalues of the operator $L = L_0 + K$ in the complement of a ball with radius

$s > \|L_0\|$ are estimated in terms of the approximation numbers and the below resolvent bound

$$\|R_{L_0}(\lambda)\| \leq \frac{1}{|\lambda| - \|L_0\|} \text{ for all } |\lambda| > \|L_0\|. \tag{9.1}$$

But it is more interesting and also natural to enlarge this region up to the complement of a ball with radius $\mathrm{spr}(L_0) := \sup\{|\lambda| : \lambda \in \sigma(L_0)\}$ (the spectral radius). But to do this it is necessary to replace (9.1) by

$$\|R_{L_0}(\lambda)\| \leq \frac{1}{|\lambda| - \mathrm{spr}(L_0)} \text{ for all } |\lambda| > \mathrm{spr}(L_0). \tag{9.2}$$

Of course (9.2) cannot be varified for all operators. In this context one should analyze the followig questions:

(i) Classify the operators A for which one can find an $M \geq 1$ and $m \in \mathbb{N}$ such that

$$\|R_A(\lambda)\| \leq \frac{M}{(|\lambda| - \mathrm{spr}(A))^m} \text{ for all } |\lambda| > \mathrm{spr}(A)$$

or

(ii) classify the operators A for which one can find an $M \geq 1$ and $m \in \mathbb{N}$ such that

$$\|R_A(\lambda)\| \leq \frac{M}{\mathrm{dist}(\lambda, \sigma(A))^m} \text{ for all } \lambda \in \rho(A).$$

(3) In Corollary 5.6 (a) we studied the spectrum of the operator

$$L = L_0 + K.$$

outside a disc with radius $\|L_0\|$. Let $p > 0$ and $K \in \mathcal{S}_p(X)$. If $q > p+1$ then

$$\sum_{\lambda \in \sigma_d(L), |\lambda| > \|L_0\|} (|\lambda| - \|L_0\|)^q < \infty. \tag{9.3}$$

According to this corollary it is natural to ask (see also [7]):

What is the infimum of $q_{\mathcal{B}}(p)$ of all exponents q such that (9.3) is true for all Banach spaces X, all $L_0 \in \mathcal{L}(X)$ and all compact operators K with $p-$summable approximation numbers, and is this infimum a minimum?

If we denote $q_{\mathcal{H}}(p)$ as the infimum of all exponents q such that (9.3) is true for all Hilbert spaces H, all $L_0 \in \mathcal{L}(H)$ and all compact operators K with $p-$summable approximation numbers, it is known (see [19] and [21]) that $q_{\mathcal{H}}(p)$ is a minimum with

$$q_{\mathcal{H}}(p) = \max(1, p).$$

This implies that we can at least enclose the quantity $q_{\mathcal{B}}(p)$ by

$$\max(1, p) \leq q_{\mathcal{B}}(p) \leq p + 1.$$

To see that at least for $p \leq 1$ the Hilbert space case is different to the Banach space case have a look to the example below. Let $L_0 = S$ be the shift operator on $l^1(\mathbb{N})$ defined by $S\delta_n = \delta_{n+1}$ ((δ_n) the canonical standard basis) and $b := (b_k) \in l^\infty(\mathbb{N})$ (see [29]) with the property that the zeros of

$$h(w) = 1 - \sum_{k=1}^{\infty} b_k w^k, \quad w \in \mathbb{D}$$

satisfy

$$\sum_{w \in \mathcal{Z}(h)} (1 - |w|) = \infty.$$

Define

$$Kf := \langle f, b \rangle \delta_1 \text{ for all } f \in l^1(\mathbb{N}),$$

then $K \in \mathcal{S}_p(l^1(\mathbb{N}))$ for all $p \leq 1$ (since K is a finite rank operator). For the eigenvalues of

$$L := S + K$$

it follows, that they coincide with the zeros of

$$\begin{aligned}
\det{}_1(\mathbb{1} - KR_S(\lambda)) &= 1 - \langle R_S(\lambda)\delta_1, b \rangle \\
&= 1 - \frac{1}{\lambda} \sum_{k=0}^{\infty} \langle S^k \delta_1, b \rangle \frac{1}{\lambda^k} \\
&= 1 - \sum_{k=1}^{\infty} b_k \frac{1}{\lambda^k} = h\left(\frac{1}{\lambda}\right).
\end{aligned}$$

By the assumption on h it follows

$$\sum_{\lambda \in \sigma_d(L) \cap \overline{\mathbb{D}}^c} (|\lambda| - \|S\|) = \sum_{\frac{1}{\lambda} \in \mathcal{Z}(h)} \frac{1 - \left|\frac{1}{\lambda}\right|}{\left|\frac{1}{\lambda}\right|} = \infty.$$

(4) In Chapter 8 regularized determinants of infinite order were discussed. They were defined for operators on finite dimensional spaces.
It seems to be fruitful to generalize Theorem 8.4, Corollary 8.5 and Corollary 8.7 to infinite dimensional Banach spaces.
One first step into this direction would be to construct a consistent definition of the term "exceptional point" in Banach spaces.

Appendix

Supplement to Section 5.2.1

In Section 5.2.1 under Assumption 5.8 Theorem 5.10 was proven.

Let $L = L_0 + K$ satisfy Assumption 5.8 and let $E \subseteq [a, b]$ be an open set (open according to $\mathbb{R}$). If the operator valued map $\rho(L_0) \ni \lambda \mapsto KR_{L_0}(\lambda)$ can be continuously extended to $E \cup \rho(L_0)$ (with respect to $\|\cdot\|_{\mathfrak{B}}$) then $E \cap \overline{\sigma_d(L)} = \emptyset$.
If $\lambda_0 \in [a, b]$ is a single point with the property that there is a continuous extension of $KR_{L_0}(\cdot)$ (with respect to $\|\cdot\|_{\mathfrak{B}}$), let us call this continuation K_{λ_0}, and 1 is not a discrete eigenvalue of K_{λ_0}, then λ_0 is not an accumulation point of $\sigma_d(L)$.

After some helpful discussions with Marcel Hansmann the author noticed that the preconditions concerning the perturbing operator K can be weakended, such that K is just a compact operator. Therefore one can modify Theorem 5.10 in the following manner .

Theorem 9.1 *Let $L = L_0 + K$, where L_0 is a bounded operator with $\sigma(L_0) = [a, b] \subseteq \mathbb{R}$ and let $E \subseteq [a, b]$ be an open set (open according to $\mathbb{R}$). If the operator valued map $\rho(L_0) \ni \lambda \mapsto KR_{L_0}(\lambda)$ can be continuously extended to $E \cup \rho(L_0)$ (with respect to the operator norm) then $E \cap \overline{\sigma_d(L)} = \emptyset$.*
If $\lambda_0 \in [a, b]$ is a single point with the property that there is a continuous extension of $KR_{L_0}(\cdot)$, let us call this continuation K_{λ_0}, and 1 is not a discrete eigenvalue of K_{λ_0}, then λ_0 is not an accumulation point of $\sigma_d(L)$.

Proof: There exists an analytic compact valued function

$$(\mathbb{C} \setminus [a, b]) \cup E \ni \lambda \mapsto K(\lambda)$$

with

$$K(\lambda) = KR_{L_0}(\lambda), \text{ for all } \lambda \in \mathbb{C} \setminus [a, b].$$

According to Theorem 1.2, for each $\lambda \in (\mathbb{C} \setminus [a,b]) \cup E$ the operator

$$\mathbb{1} - K(\lambda)$$

is a Fredholm operator (the invertible operator $\mathbb{1}$ is Fredholm).
Therefore

$$(\mathbb{C} \setminus [a,b]) \cup E \ni \lambda \mapsto \mathbb{1} - K(\lambda)$$

is an analytic Fredhom-valued map.
Due to [14] Theorem XI.8.4 the set

$$\Sigma := \{\lambda \in (\mathbb{C} \setminus [a,b]) \cup E : \mathbb{1} - K(\lambda) \text{ is not invertible}\}$$

is at most countable and has no accumulation points inside Σ. Thus $1 \in \sigma(K(\lambda))$ if and only if $\lambda \in \Sigma$. But $\lambda \in \mathbb{C} \setminus [a,b]$ is a discrete eigenvalue of L if and only if 1 is an eigenvalue of $KR_{L_0}(\lambda) = K(\lambda)$, hence $\sigma_d(L) \subseteq \Sigma$ and $\sigma_d(L)$ has no accumulation points in E.
If λ_0 is a single point with the property that there is a compact operator K_{λ_0} with $1 \notin \sigma(K_{\lambda_0})$ and $\|K_{\lambda_0} - KR_{L_0}(\lambda)\| \overset{\lambda \to \lambda_0}{\to} 0$ then due to Theorem 1.12 there has to be an $\epsilon > 0$ such that $1 \notin \sigma\left(KR_{L_0}(\lambda)\right)$ for all λ with $|\lambda - \lambda_0| < \epsilon$. This implies $\lambda \notin \sigma_d(L)$ for all λ with $|\lambda - \lambda_0| < \epsilon$. $\square$

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